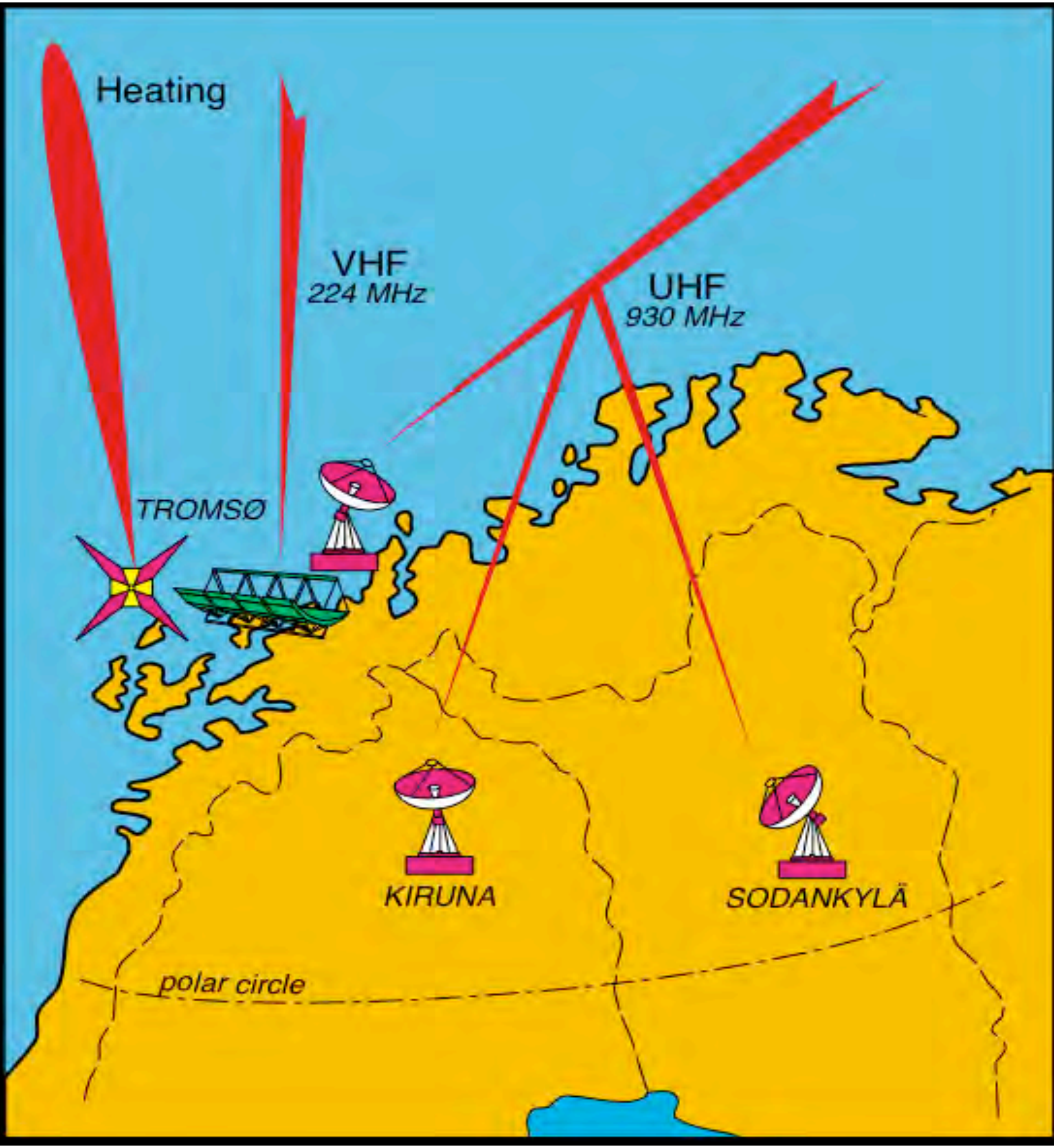
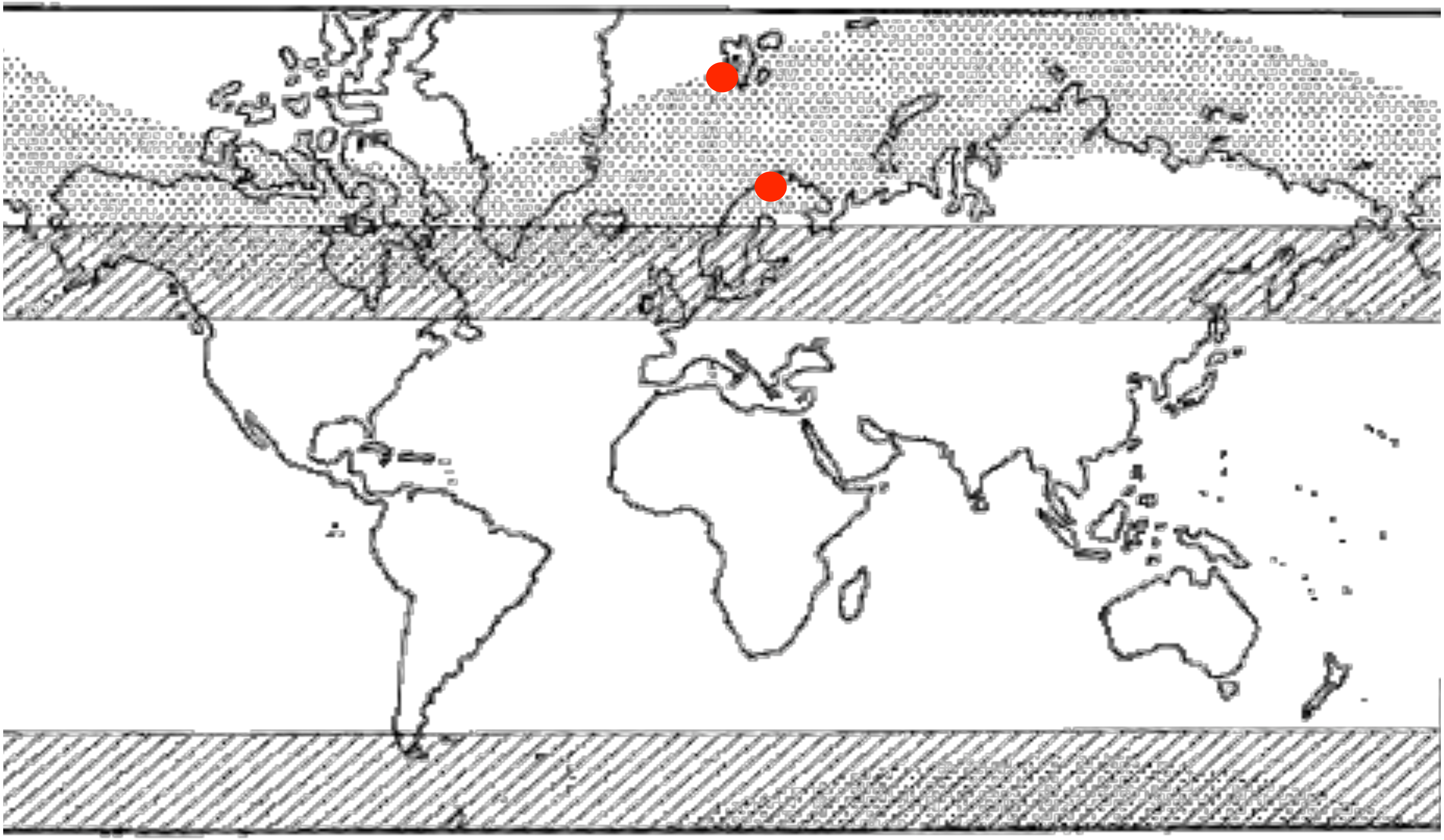
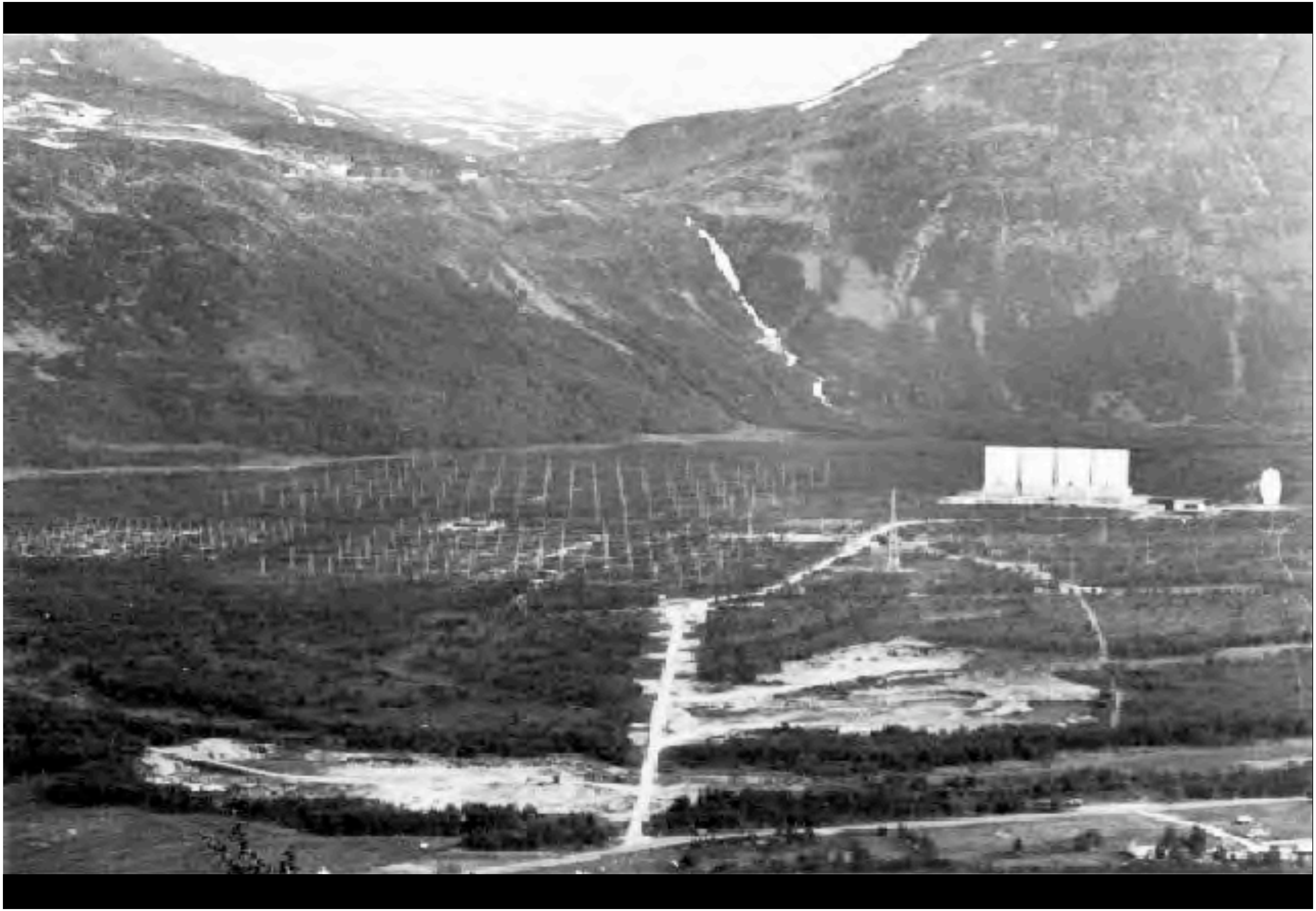


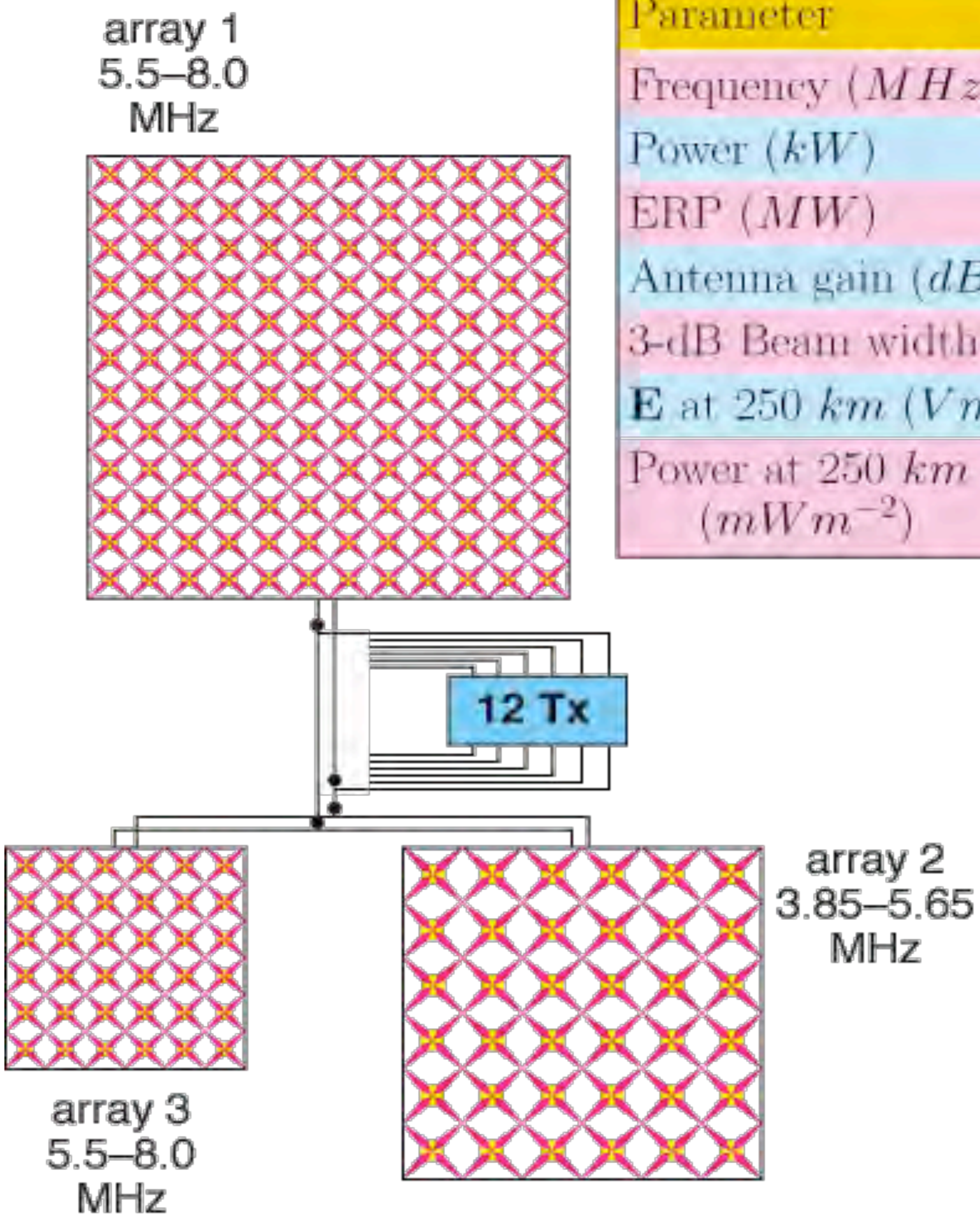
EISCAT Radar School Kiruna 2005

by

Cesar La Hoz





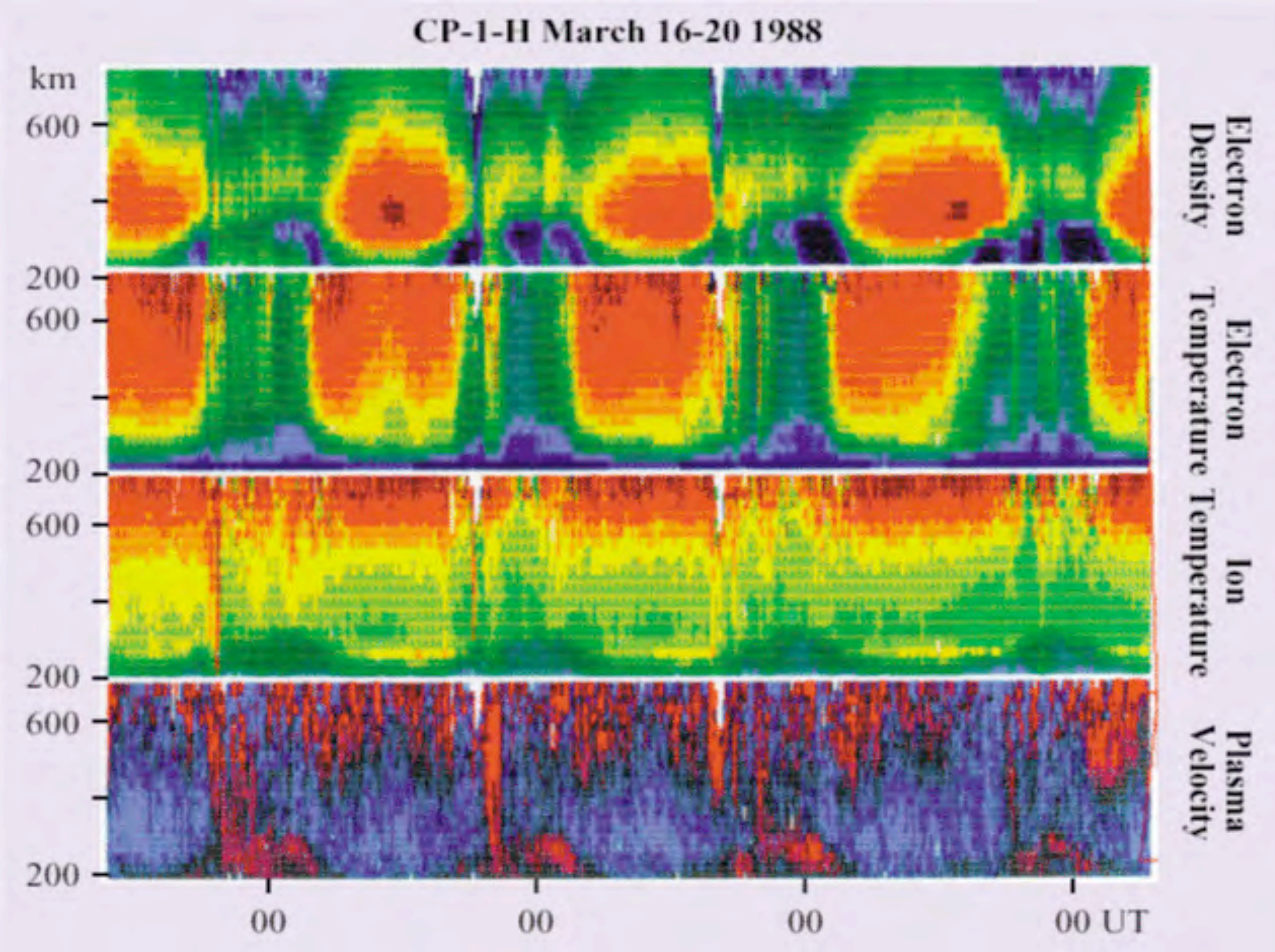


Parameter	Trans.	Array 1	Array 2	Array
Frequency (MHz)	2.7 – 8.0	5.5 – 8.0	4.0 – 5.5	5.5 – 8.0
Power (kW)	12×100			
ERP (MW)		1200	300	300
Antenna gain (dB)		30	24	24
3-dB Beam width		7.5°	14.5°	14.5°
$\mathbf{E}$ at 250 km (Vm^{-1})		1	0.5	0.5
Power at 250 km (mWm^{-2})		1.6	0.4	0.4

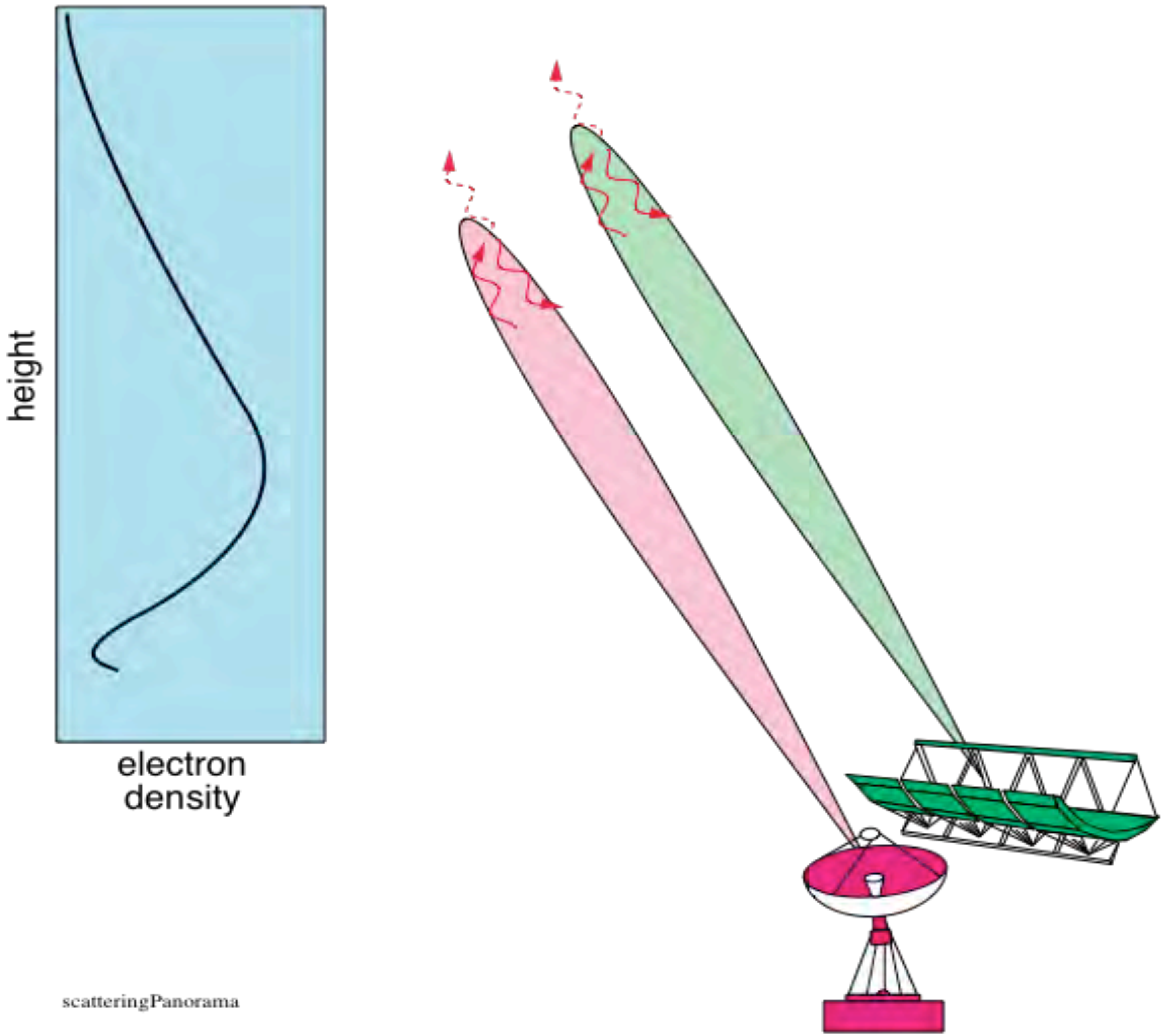
The Tromsø Heating Facility

HeatingSchematic

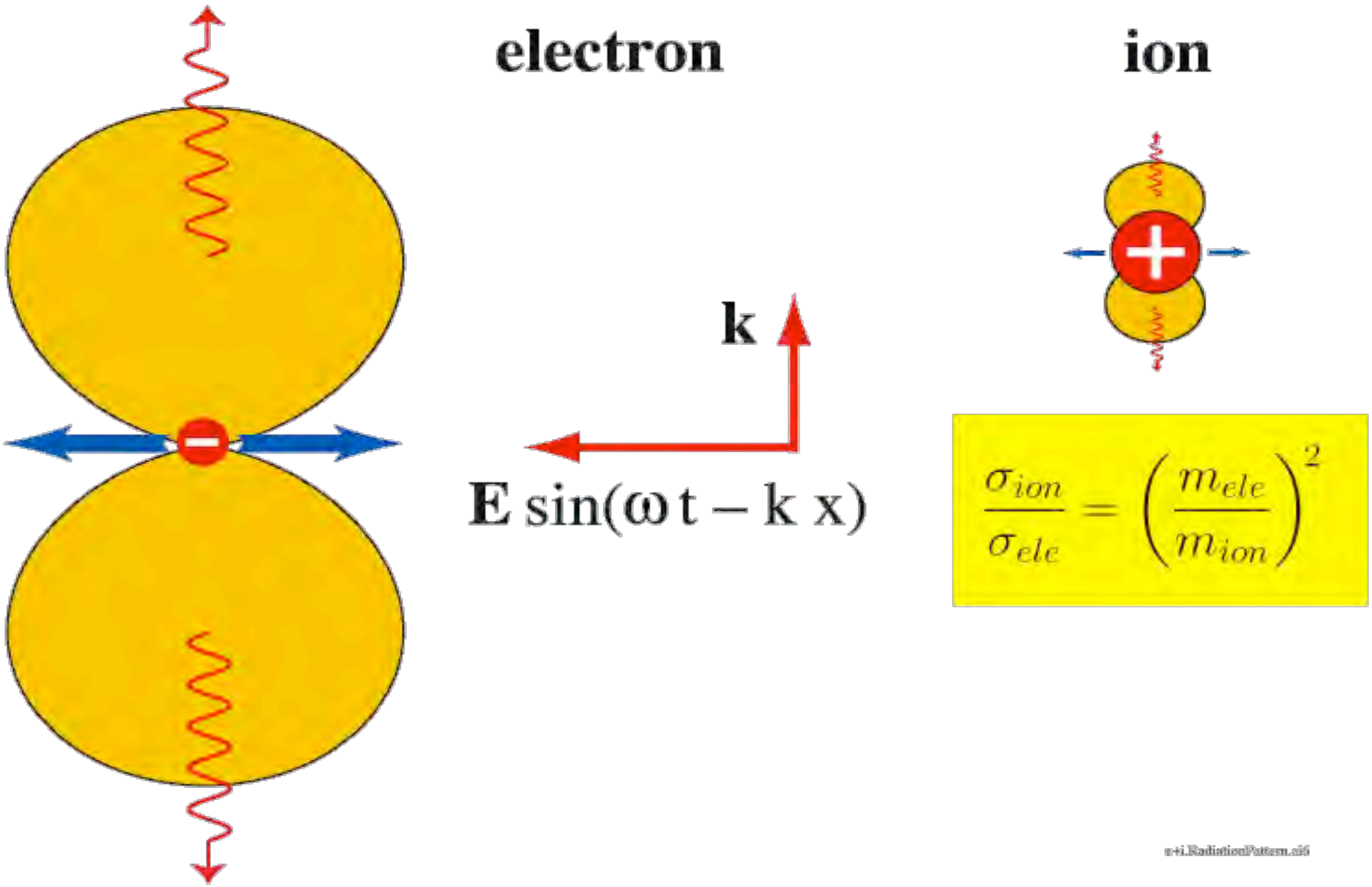

Photo: Tony Van Eyken



CP-1-H March 16-20 1988
km
600
200
600
200
600
200
600
200
00
00
00
00 UT
Electron Density
Electron Temperature
Ion Temperature
Plasma Velocity



scatteringPanorama


electron
ion
k
$\mathbf{E}\sin(\omega t - k x)$
$$\frac{\sigma_{ion}}{\sigma_{ele}} = \left(\frac{m_{ele}}{m_{ion}}\right)^2$$
e+i.RadiationPattern.ai6

AVIATION WEEK
& SPACE TECHNOLOGY

A McGRAW-HILL PUBLICATION $5.00

APRIL 9, 1990

AIR FORCE F-117A MOVES OUT OF THE BLACK

PAGE 36

NORTHWEST AIRLINES: DEVELOPING NEW IMAGE

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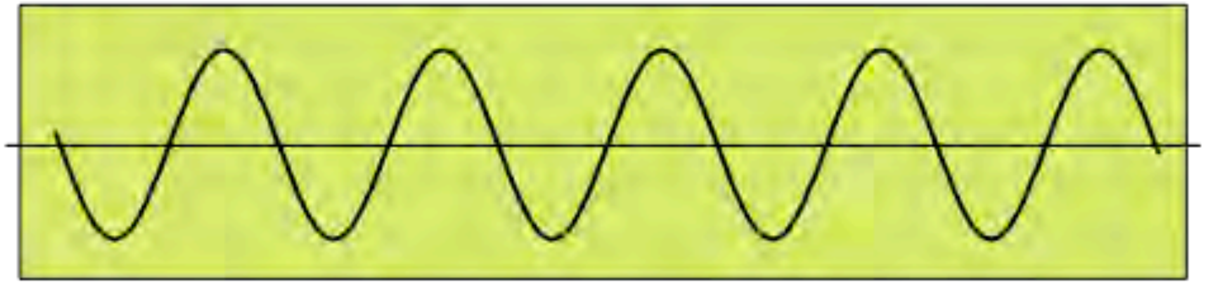
SOVIET MILITARY SPACE: MODERNIZING ORBITAL ASSETS

PAGE 44

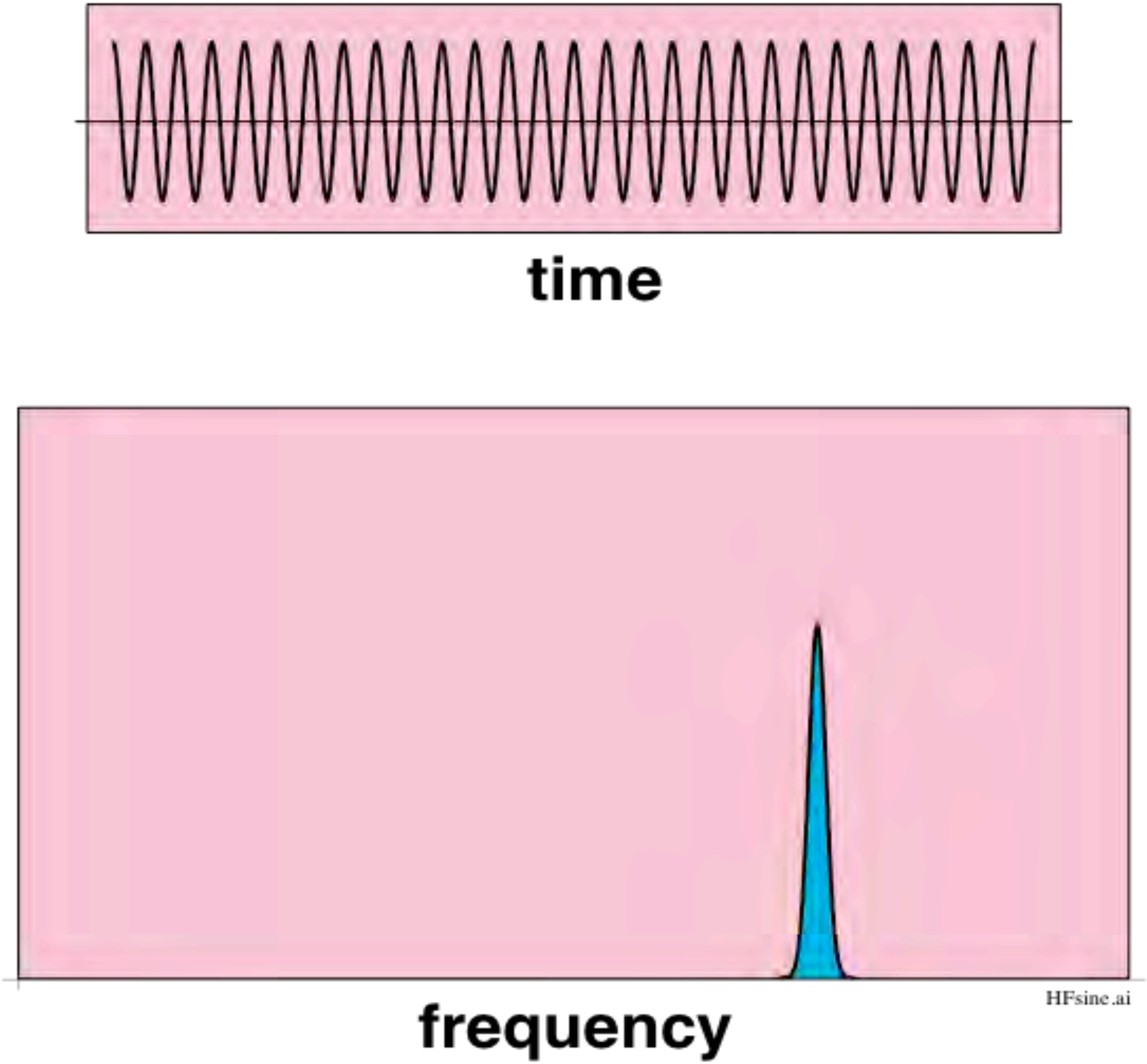

SWAMP

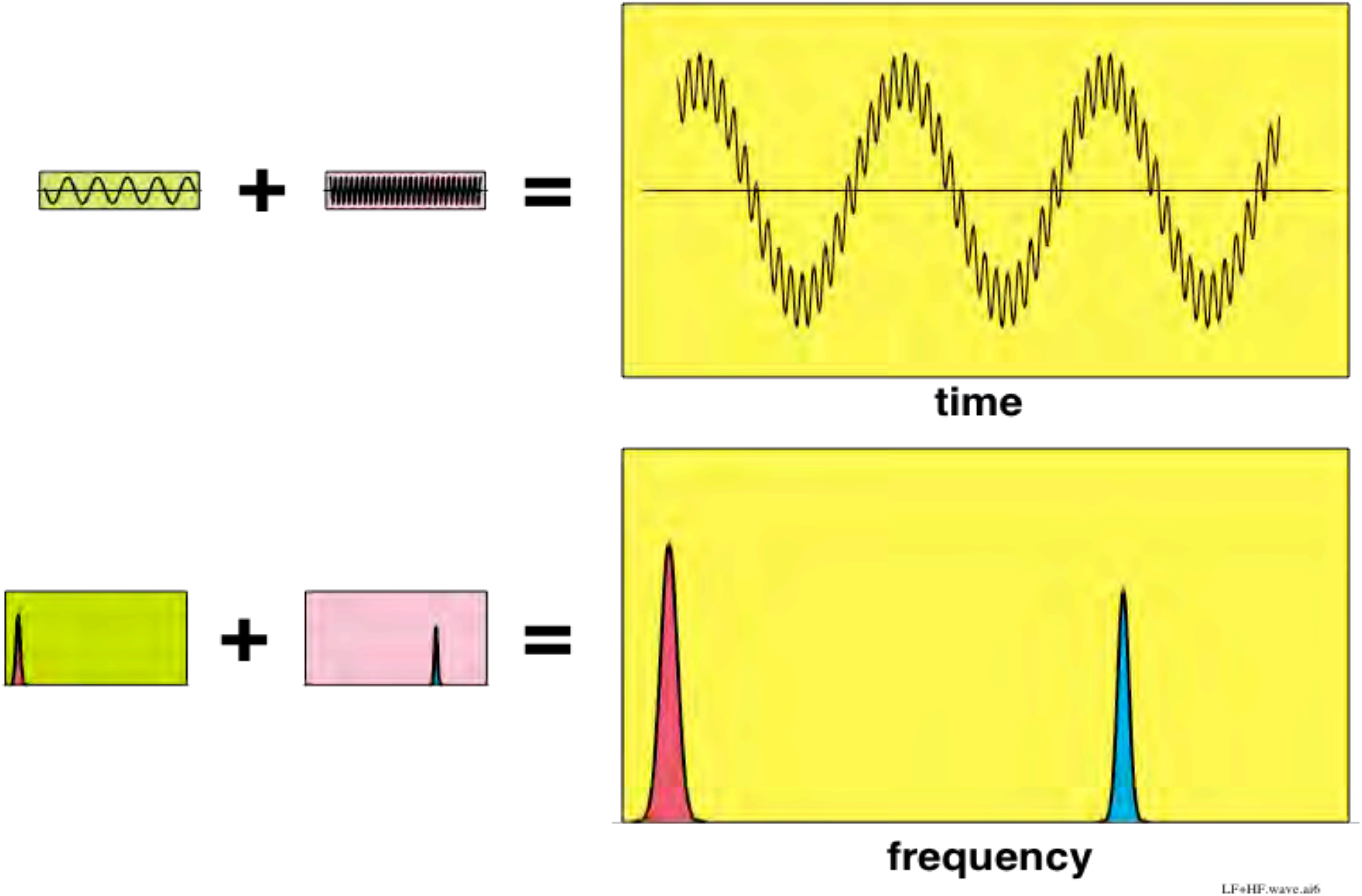
swamp.ai6



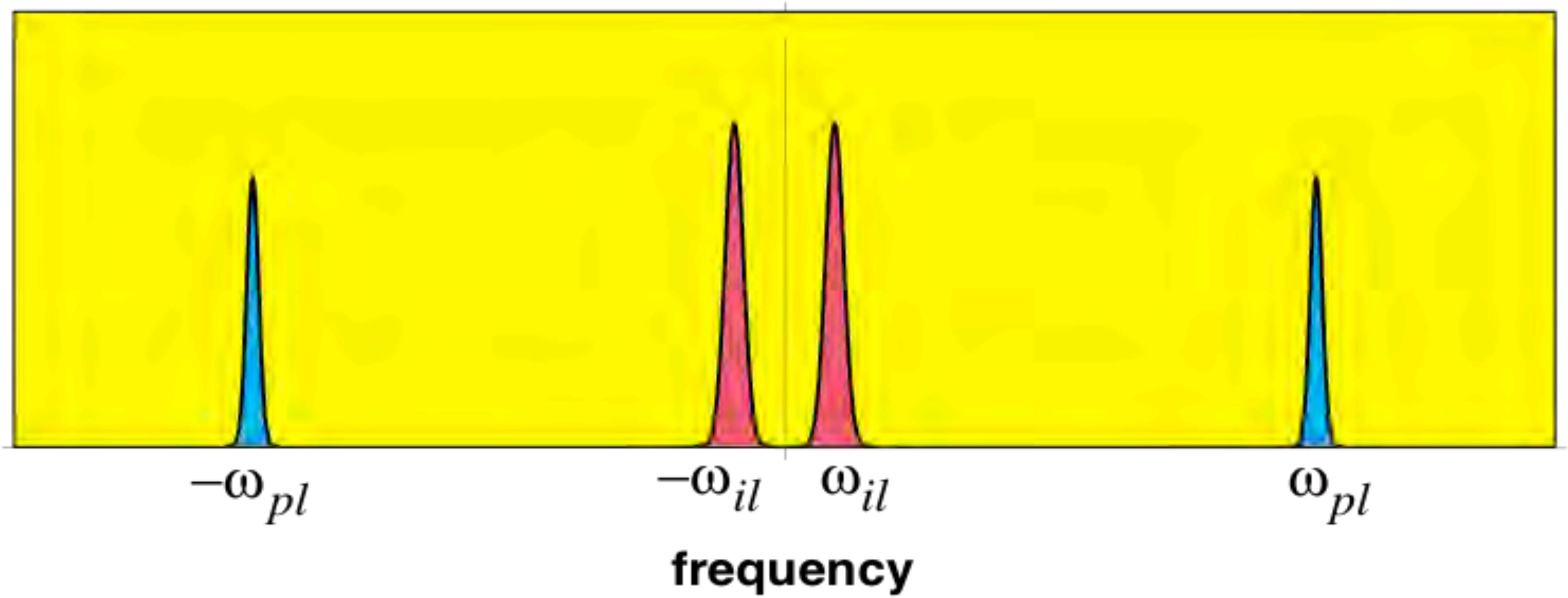
time

frequency


time
HFsine.ai
frequency

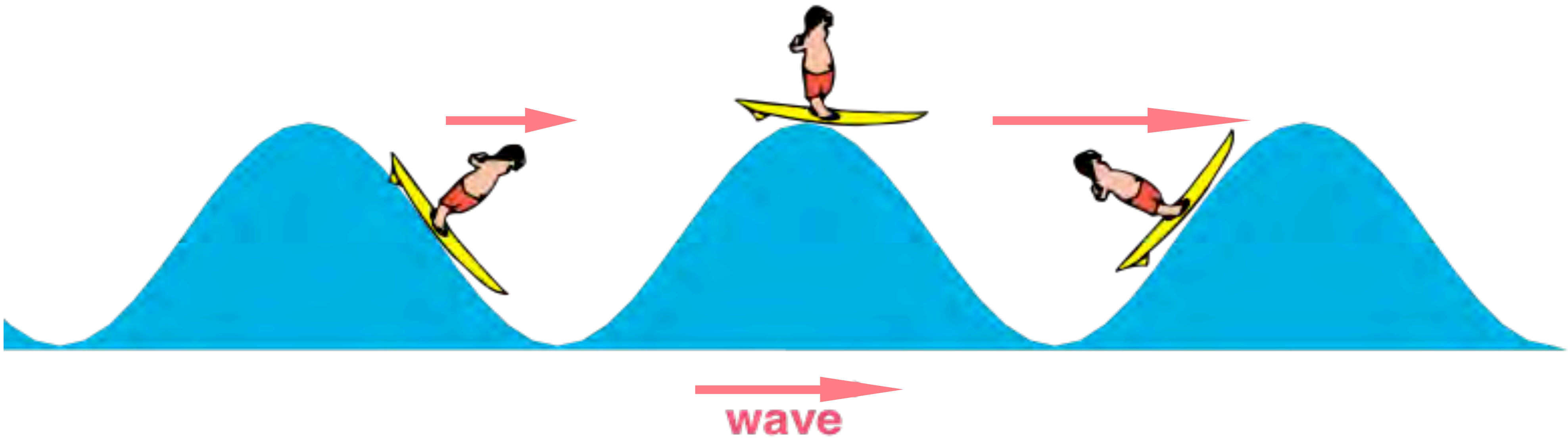


+
=
time
+
=
frequency
LF+HF.wave.ai6



PL+IL.no.damping.ai6

Landau wave-particle interactions

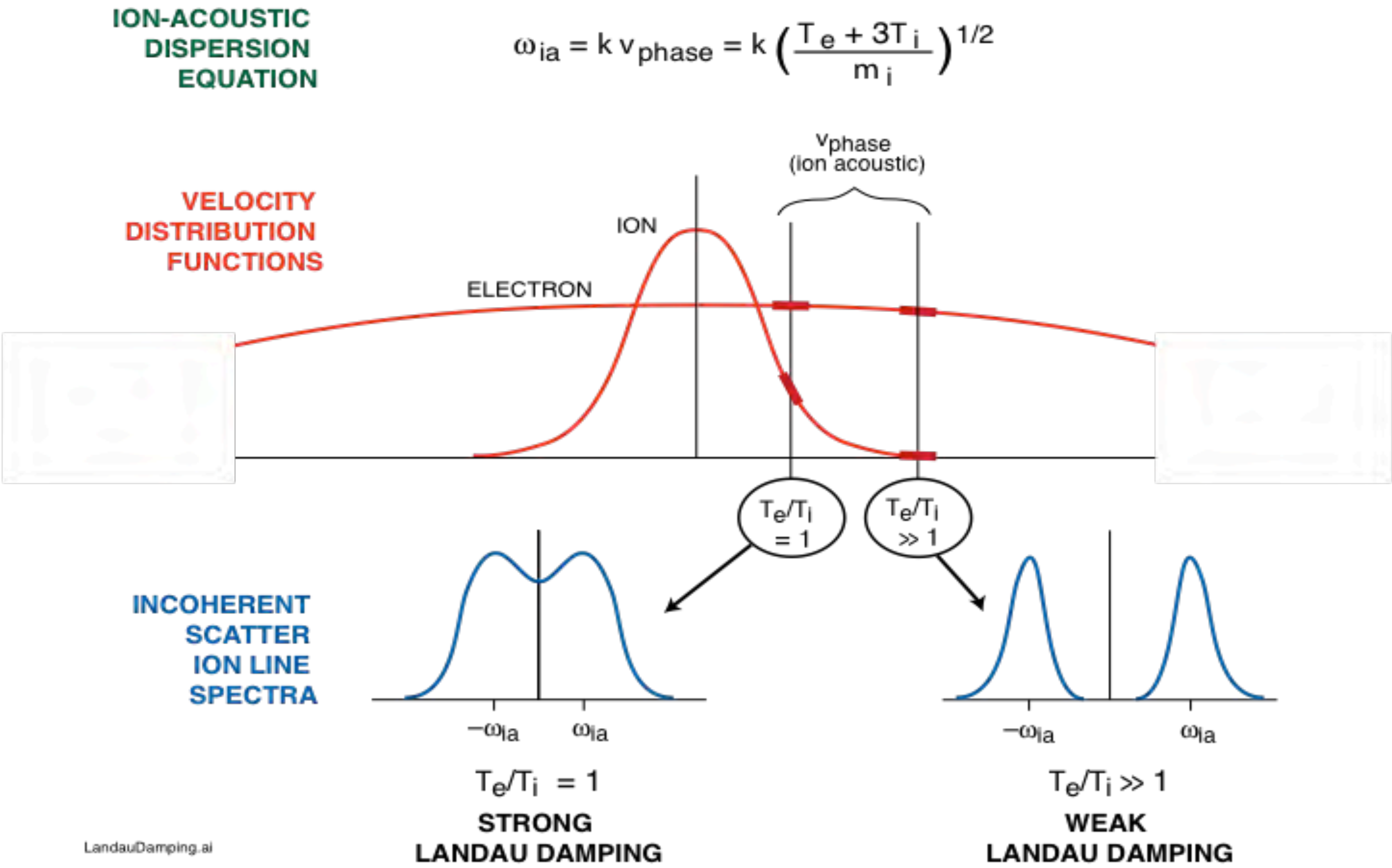


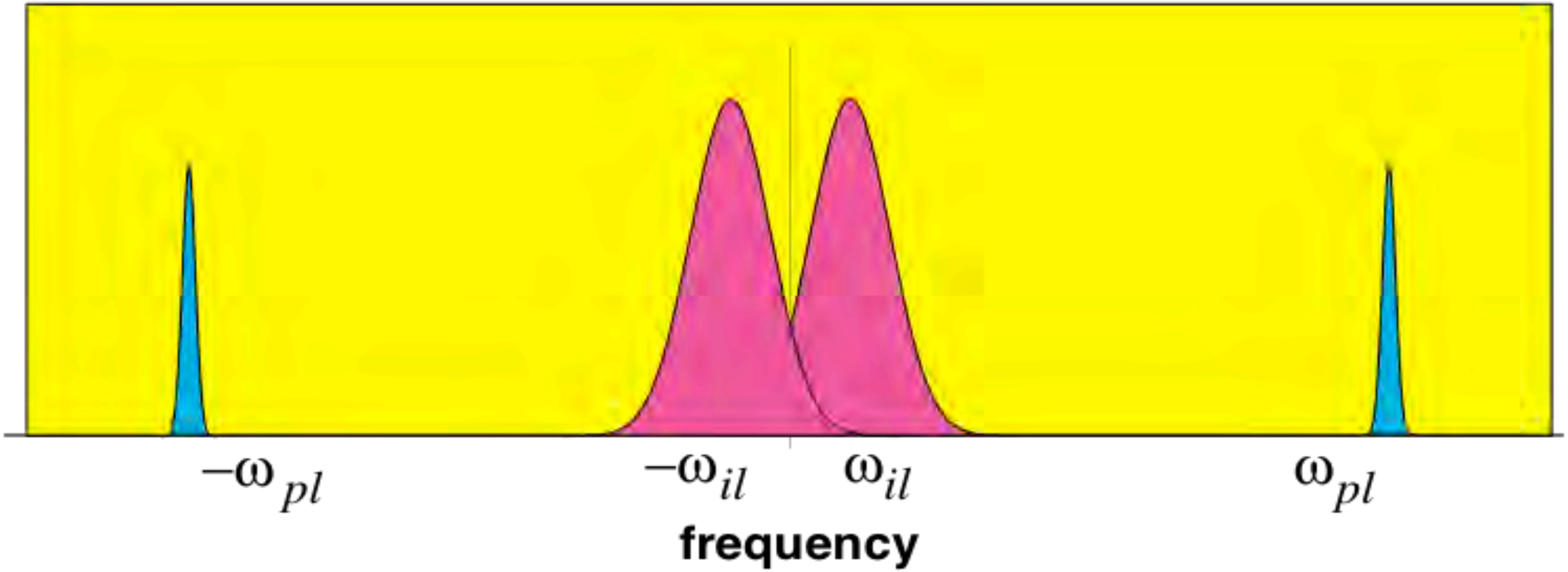
particle gains energy

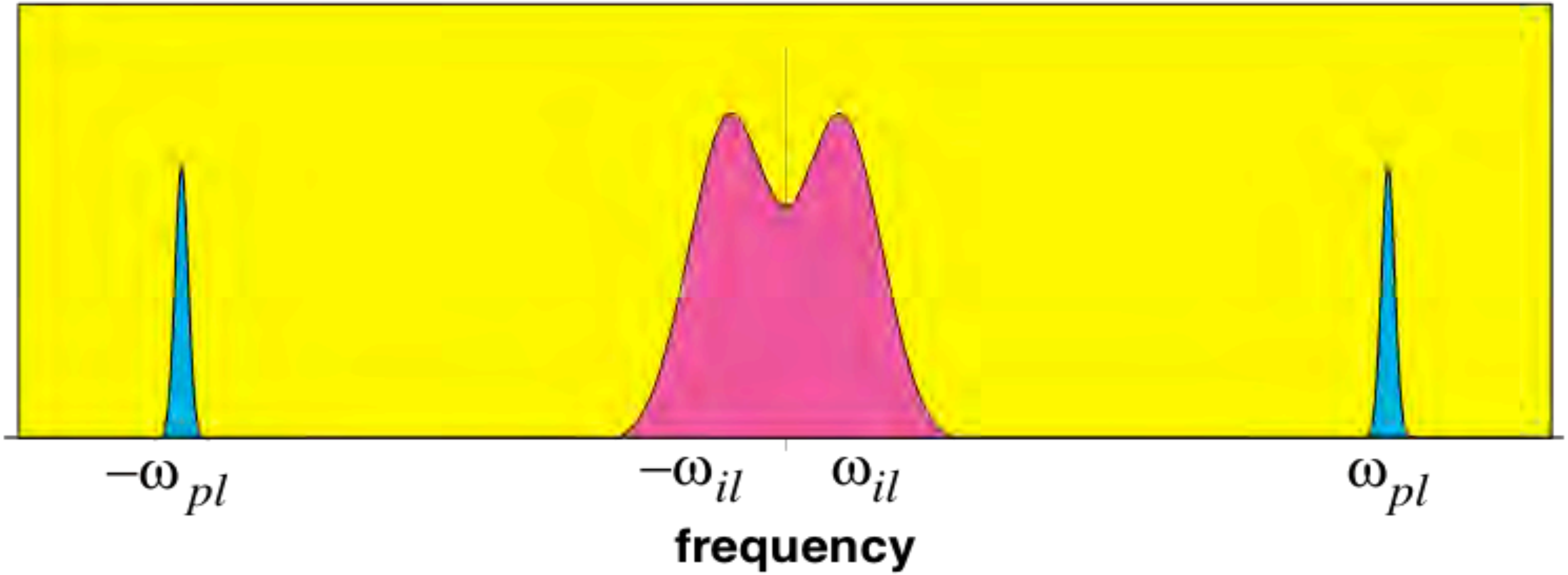
wave gains energy

Landau.surfer.ai6

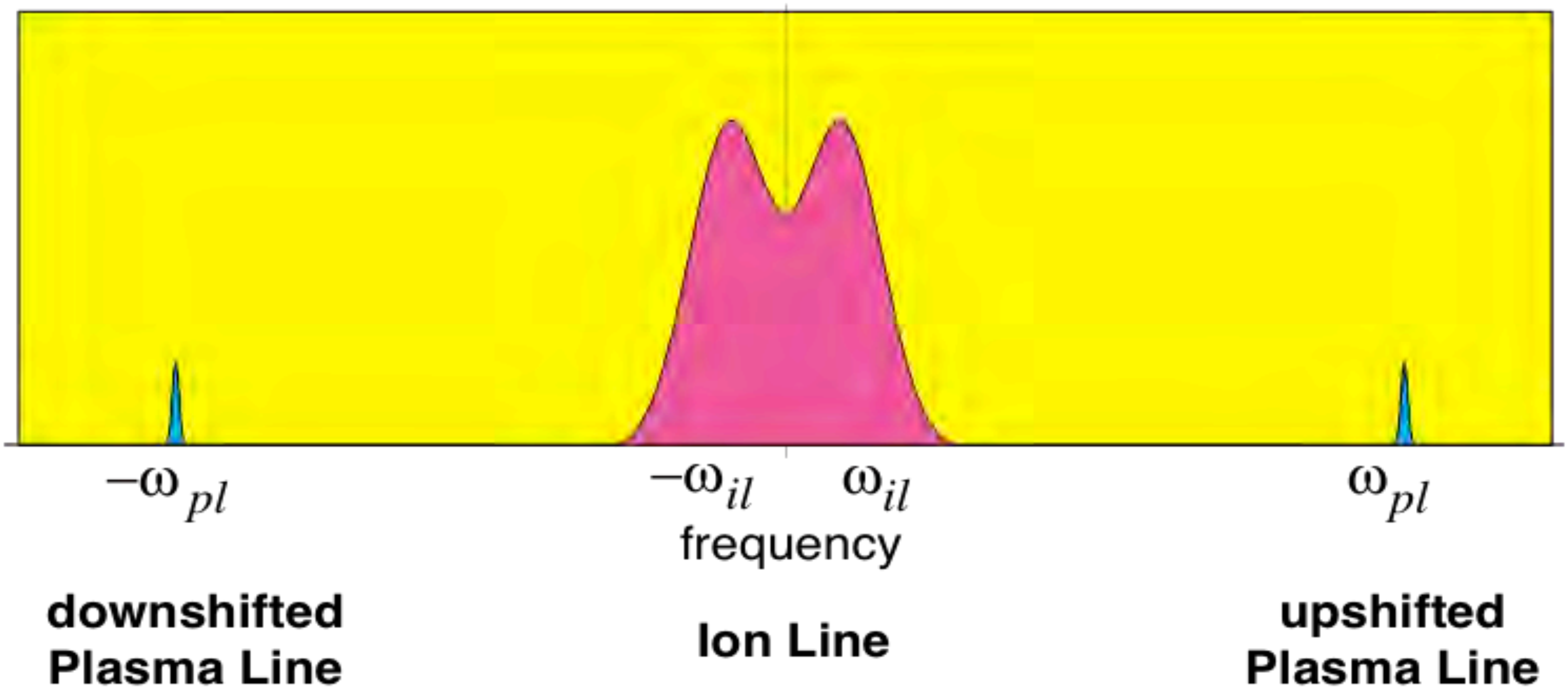
THE EFFECT OF LANDAU DAMPING ON THE INCOHERENT SCATTER ION LINE SPECTRUM




$-\omega_{pl}$
$-\omega_{il}$
ω_{il}
ω_{pl}
frequency



PL+IL with damping 2



PL+IL.with.damping3.ai6

Incoherent Scattering Spectrum

$$S_e(\mathbf{k},\omega) = N_e \left|1 - \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}\right|^2 \int d\mathbf{v}\, f_e(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v}) + N_i \left|\frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}\right|^2 \int d\mathbf{v}\, f_i(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v})$$

electric susceptibility	$\chi_{e,i}(\mathbf{k},\omega)$
dielectric constant function	$\epsilon(\mathbf{k},\omega)$
velocity distribution function	$f_{e,i}(\mathbf{v})$

$$\epsilon(\mathbf{k},\omega) = 1 + \sum_\alpha \chi_\alpha(\mathbf{k},\omega)$$

$$\boxed{\chi_q(\mathbf{k},\omega) = \frac{\omega_{pq}^2}{k^2}\int \frac{\mathbf{k}\cdot\partial_v f_o(\mathbf{v})}{\omega - \mathbf{k}\cdot\mathbf{v}} d^3\mathbf{v}}$$

IS.SpecFormula1.ai6

Plasma Line $S_{PL}(\mathbf{k},\omega)$ **Ion Line** $S_{IL}(\mathbf{k},\omega)$

$$S_e(\mathbf{k},\omega) = N_e\left|1-\frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}\right|^2\int d\mathbf{v}\,f_e(\mathbf{v})\delta(\omega-\mathbf{k}\cdot\mathbf{v}) + N_i\left|\frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}\right|^2\int d\mathbf{v}\,f_i(\mathbf{v})\delta(\omega-\mathbf{k}\cdot\mathbf{v})$$

Plasma Line $S_{PL}(\mathbf{k},\omega)$ **Ion Line** $S_{IL}(\mathbf{k},\omega)$

$$S_e(\mathbf{k},\omega) = N_e\left|1-\frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}\right|^2\int d\mathbf{v}\, f_e(\mathbf{v})\delta(\omega-\mathbf{k}\cdot\mathbf{v}) + N_i\left|\frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}\right|^2\int d\mathbf{v}\, f_i(\mathbf{v})\delta(\omega-\mathbf{k}\cdot\mathbf{v})$$

$$\epsilon(\mathbf{k},\omega) = \mathbf{0}$$

$$\omega_{pl}(k) \approx \omega_{pe}(1+3\lambda_D^2k^2)$$

$$\omega_{ia}(k) \approx k\sqrt{\frac{T_e+3T_i}{m_i}}$$

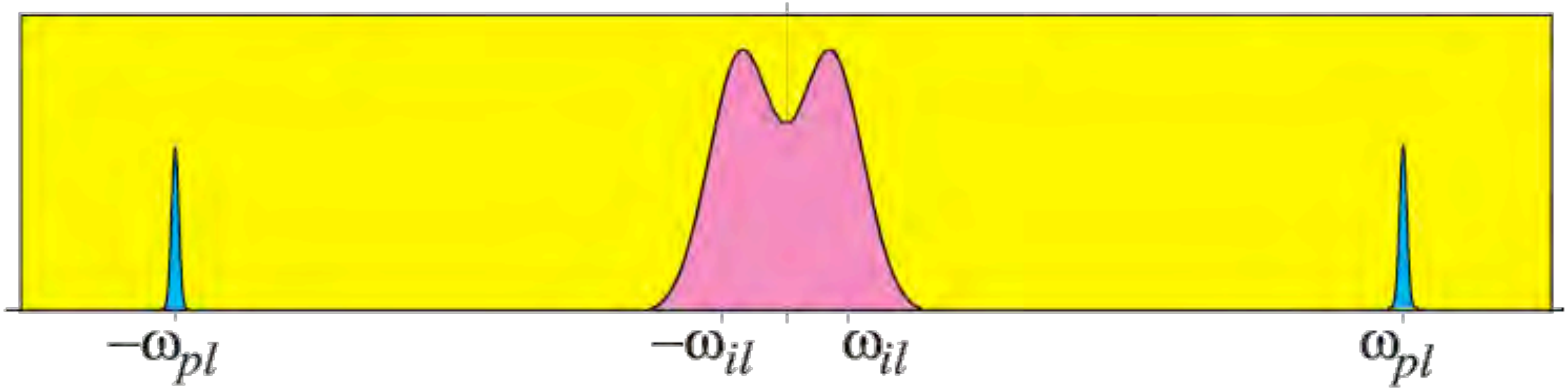
Plasma Line $S_{PL}(\mathbf{k},\omega)$ **Ion Line** $S_{IL}(\mathbf{k},\omega)$

$$S_e(\mathbf{k},\omega) = N_e \left| 1 - \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)} \right|^2 \int d\mathbf{v}\, f_e(\mathbf{v}) \delta(\omega - \mathbf{k}\cdot\mathbf{v}) + N_i \left| \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)} \right|^2 \int d\mathbf{v}\, f_i(\mathbf{v}) \delta(\omega - \mathbf{k}\cdot\mathbf{v})$$

$$\epsilon(\mathbf{k},\omega) = \mathbf{0}$$

$$\omega_{pl}(k) \approx \omega_{pe}(1 + 3\lambda_D^2 k^2)$$

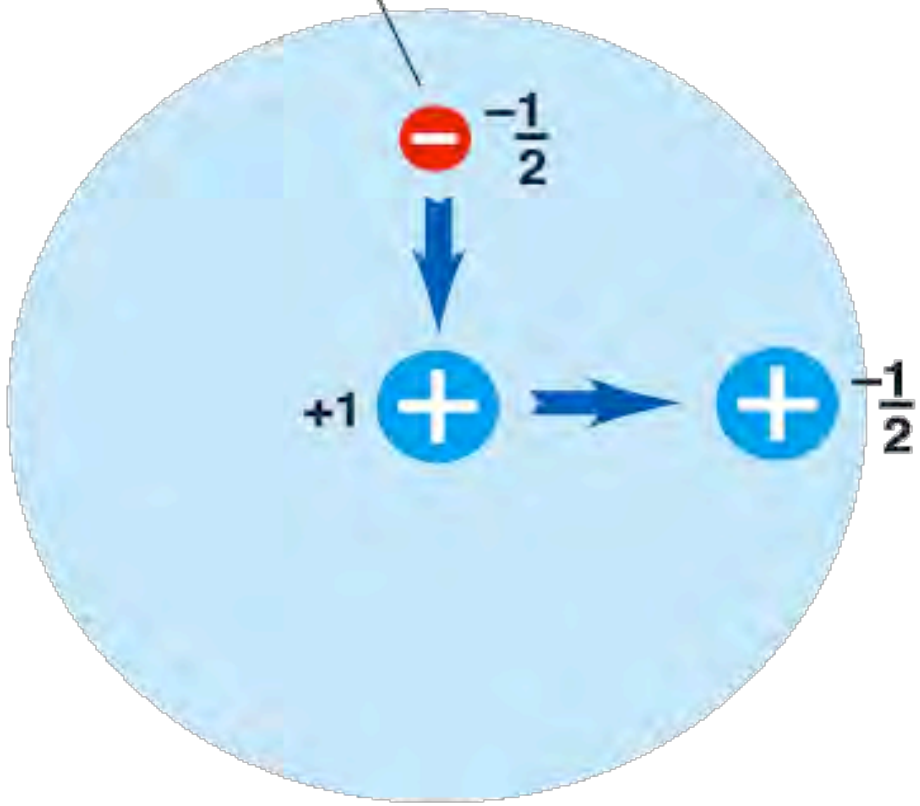
$$\omega_{ia}(k) \approx k\sqrt{\frac{T_e + 3T_i}{m_i}}$$



IS.SpecFormula4.ai6

Plasma Line $S_{PL}(\mathbf{k},\omega)$ **Ion Line** $S_{IL}(\mathbf{k},\omega)$

$$S_e(\mathbf{k},\omega) = N_e \left| 1 - \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)} \right|^2 \int d\mathbf{v}\, f_e(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v}) + N_i \left| \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)} \right|^2 \int d\mathbf{v}\, f_i(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v})$$

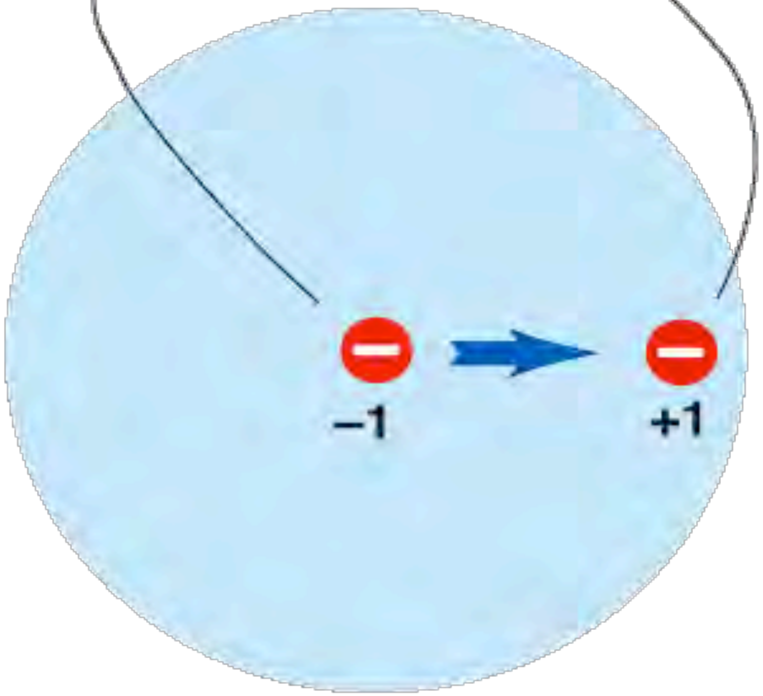


ion with cloud

IS.SpecFormula3.ai

Plasma Line $S_{PL}(\mathbf{k},\omega)$ **Ion Line** $S_{IL}(\mathbf{k},\omega)$

$$S_e(\mathbf{k},\omega) = N_e \left| 1 - \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)} \right|^2 \int d\mathbf{v}\, f_e(\mathbf{v}) \delta(\omega - \mathbf{k}\cdot\mathbf{v}) + N_i \left| \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)} \right|^2 \int d\mathbf{v}\, f_i(\mathbf{v}) \delta(\omega - \mathbf{k}\cdot\mathbf{v})$$

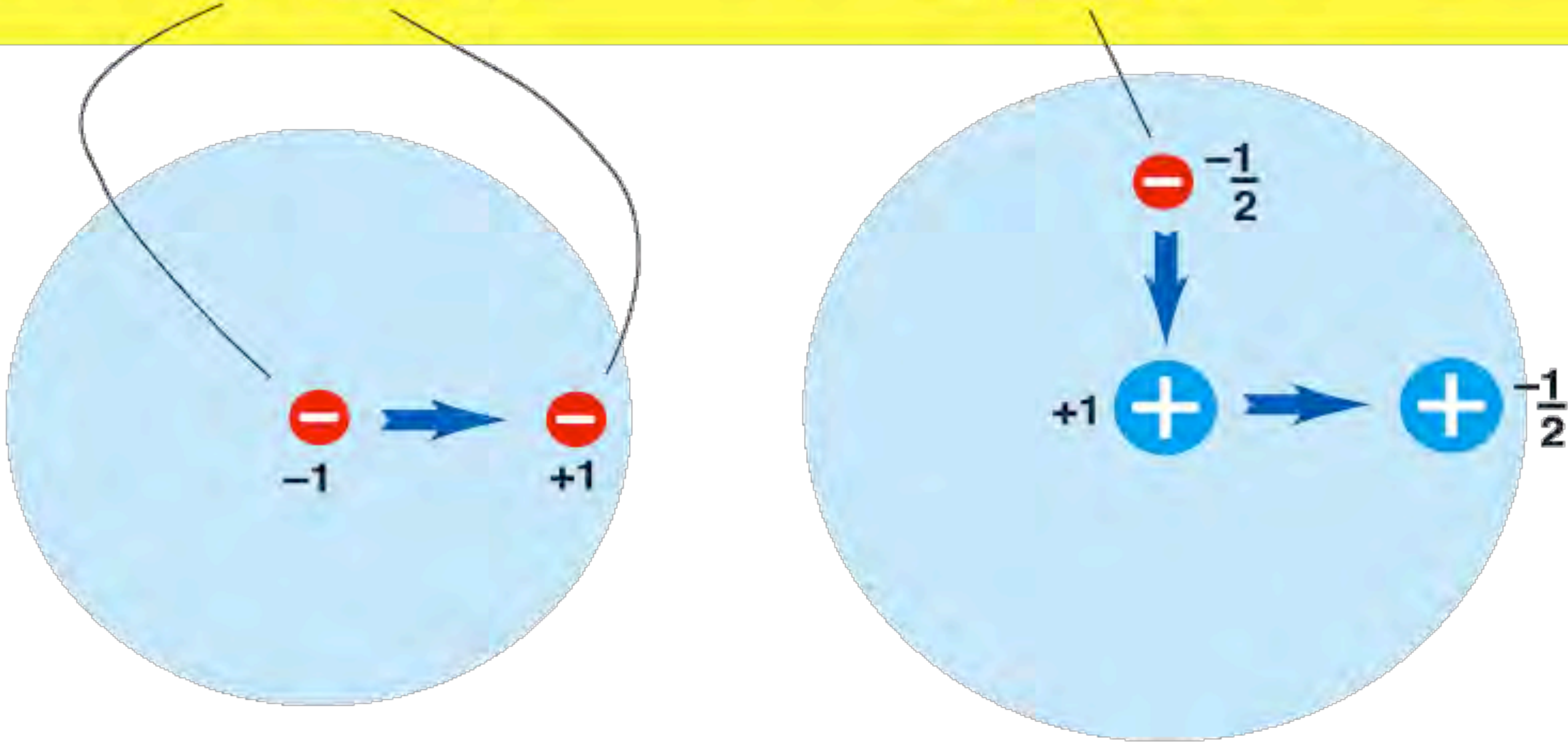


electron with cloud

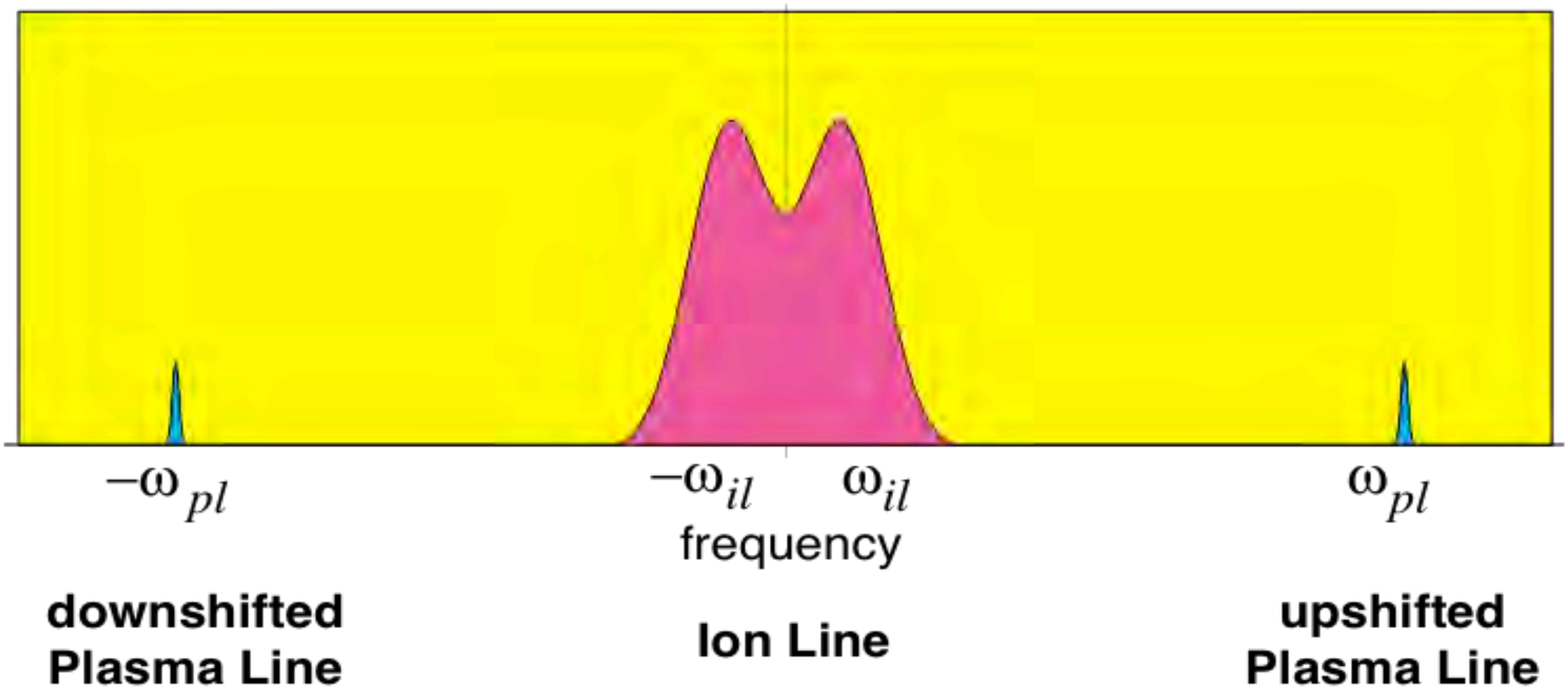
IS.SpecFormula3.ai

Plasma Line $S_{PL}(\mathbf{k},\omega)$ **Ion Line** $S_{IL}(\mathbf{k},\omega)$

$$S_e(\mathbf{k},\omega) = N_e \left| 1 - \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)} \right|^2 \int d\mathbf{v}\, f_e(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v}) + N_i \left| \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)} \right|^2 \int d\mathbf{v}\, f_i(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v})$$



electron with cloud **ion with cloud**



PL+IL.with.damping3.ai6

no collective interactions

$$S_e(\mathbf{k},\omega) = N_e \left| 1 - \cancel{\frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}} \right|^2 \int d\mathbf{v}\, f_e(\mathbf{v}) \delta(\omega - \mathbf{k}\cdot\mathbf{v}) + N_i \left| \cancel{\frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}} \right|^2 \int d\mathbf{v}\, f_i(\mathbf{v}) \delta(\omega - \mathbf{k}\cdot\mathbf{v})$$

no collective interactions

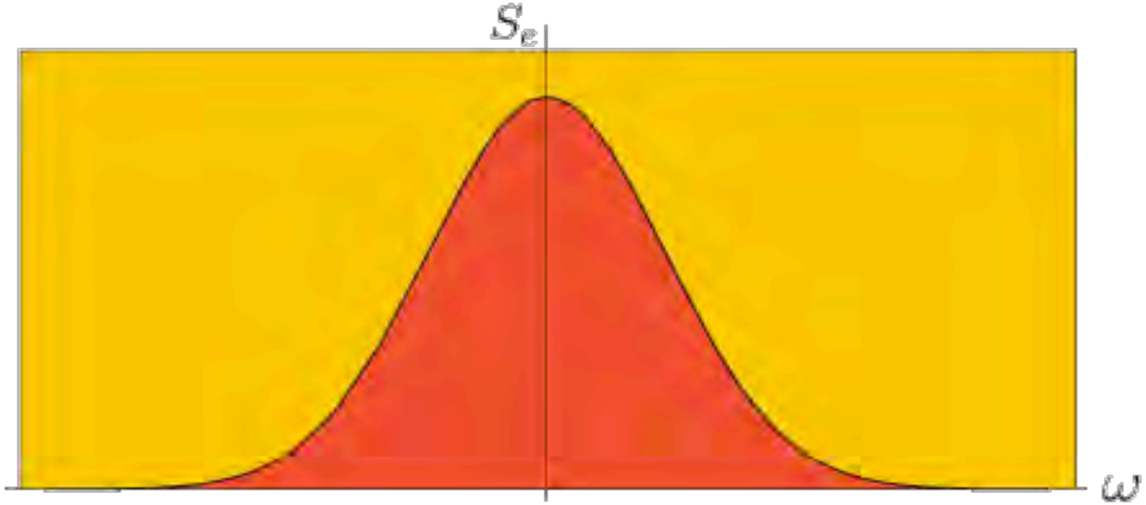
$$S_e(\mathbf{k},\omega) = N_e \left| 1 - \cancel{\frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}} \right|^2 \int d\mathbf{v}\, f_e(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v}) + N_i \left| \cancel{\frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}} \right|^2 \int d\mathbf{v}\, f_i(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v})$$

$$S_e(\mathbf{k},\omega) = N_e \int d\mathbf{v}\, f_e(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v})$$

no collective interactions

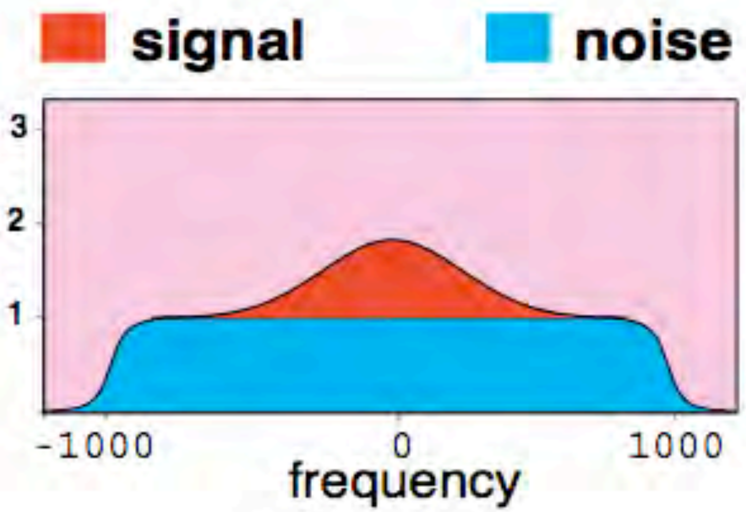
$$S_e(\mathbf{k},\omega) = N_e \left| 1 - \cancel{\frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}} \right|^2 \int d\mathbf{v}\, f_e(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v}) + N_i \left| \cancel{\frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}} \right|^2 \int d\mathbf{v}\, f_i(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v})$$

$$S_e(\mathbf{k},\omega) = N_e \int d\mathbf{v}\, f_e(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v})$$

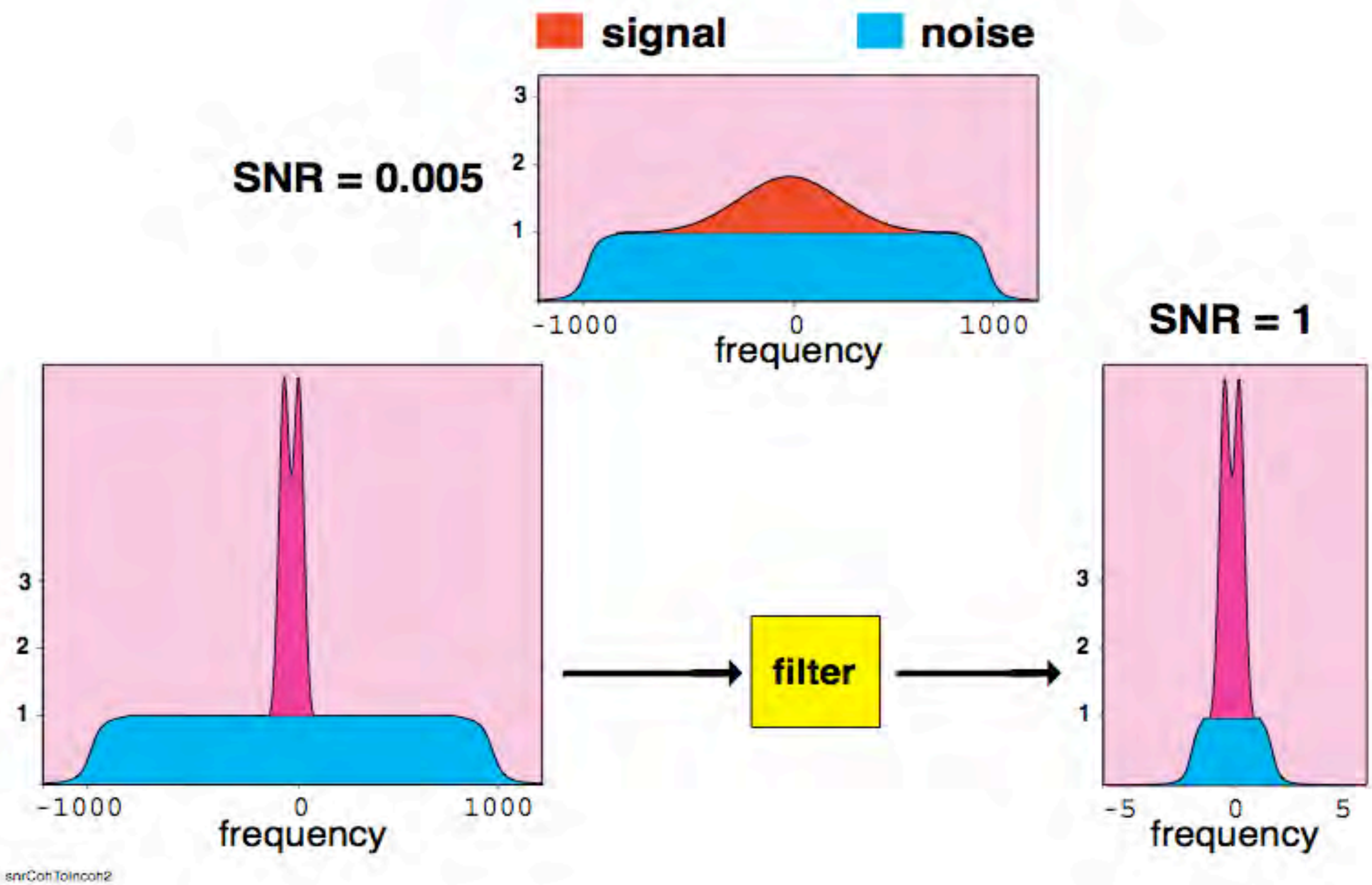


18.SpecFormula7.nb

SNR = 0.005



snrCohToIncoh1

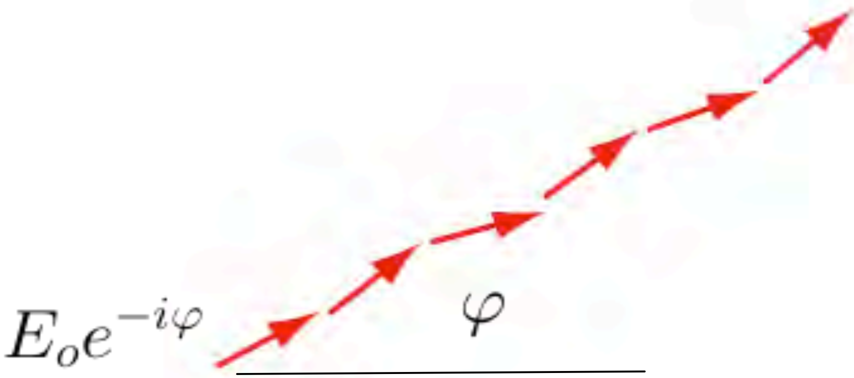

signal
noise
SNR = 0.005
3
2
1
-1000
0
1000
frequency
SNR = 1
3
2
1
-1000
0
1000
frequency
filter
3
2
1
-5
0
5
frequency
snrCohToIncoh2

Coherent Scattering

$$E = E_o e^{-i(\omega t - k x_r)}$$

$$P \propto \langle E E^* \rangle = E_o^2 \sum_{r,s}^{N} \langle e^{-ik(x_r - x_s)} \rangle = N^2 E_o^2$$

$$\varphi = k(x_r - x_s) = 0 \bmod (2\pi)$$



Incoherent Scattering

$$E = E_o e^{-i(\omega t - k x_r)}$$

$$P \propto \langle E E^* \rangle = E_o^2 \sum_{r,s}^{N} \langle e^{-ik(x_r - x_s)} \rangle$$

$$= N E_o^2 + E_o^2 \sum_{r \neq s}^{N} \langle e^{-ik(x_r - x_s)} \rangle = N E_o^2$$

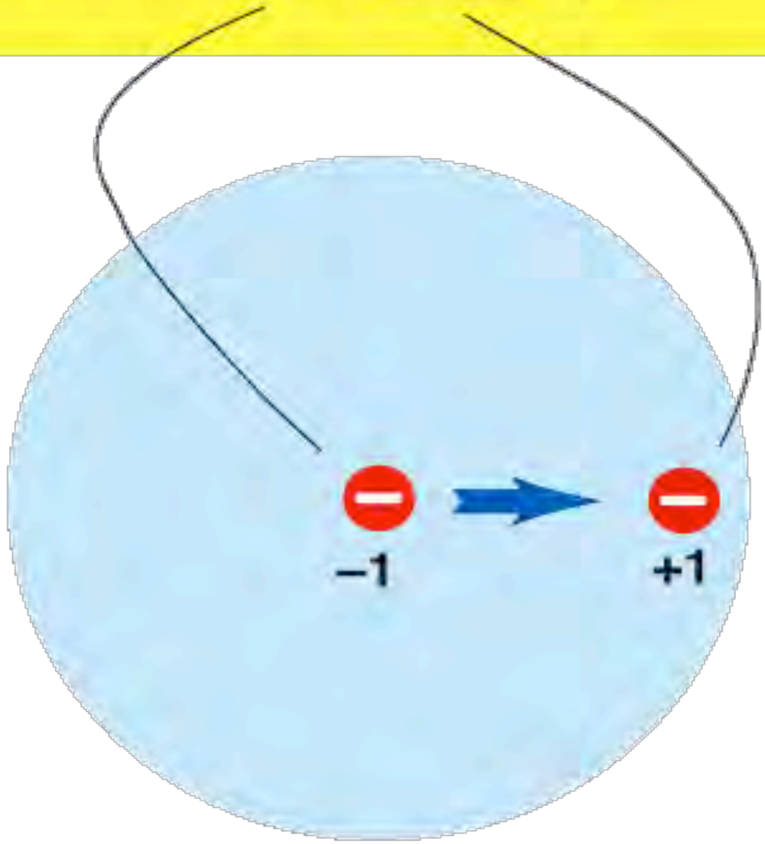
$\sum_j$  = 0

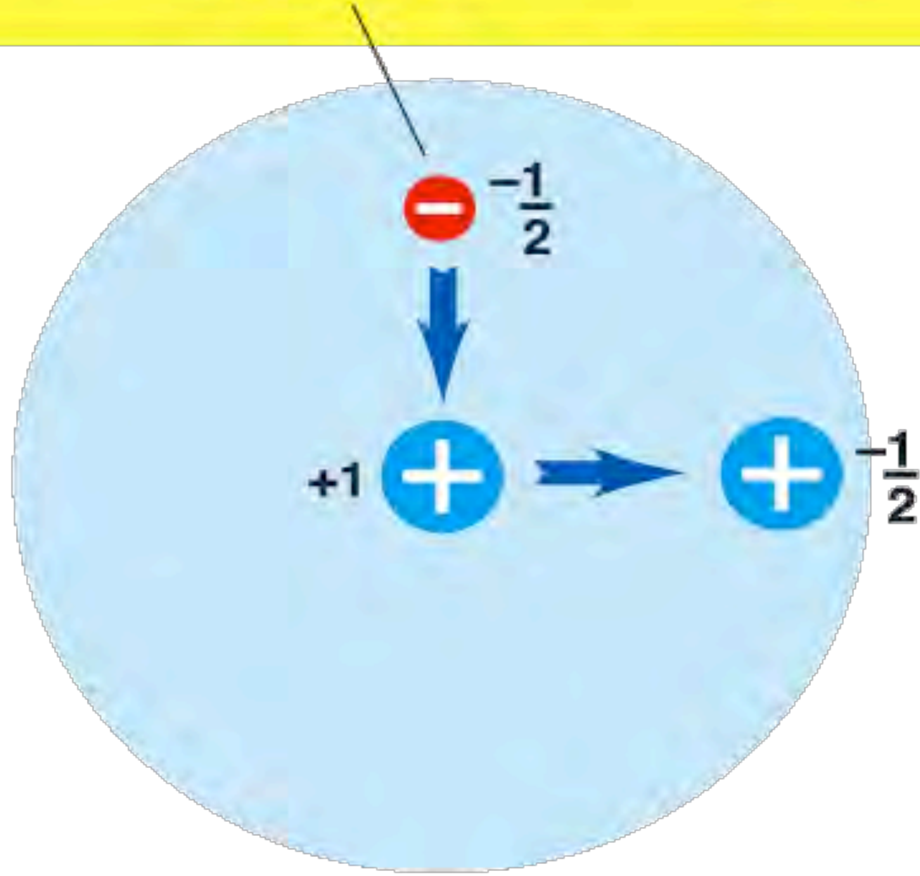
Plasma Linear Response Functions

Free electron E-field	$\mathbf{E}$
Induced E-field	$\mathbf{E}_i = \chi \mathbf{E}$
Total E-field	$\mathbf{E}_T = \mathbf{E} + \chi \mathbf{E}$
	$= \mathbf{E}(1+\chi)$
	$= \epsilon \mathbf{E}$
Susceptibility	$\chi_\alpha(\mathbf{k},\omega) = \frac{\omega_{pe}^2}{k^2} \int_{\mathcal{L}} \frac{\mathbf{k}\cdot\partial_{\mathbf{v}} f_\alpha(\mathbf{v})}{\omega - \mathbf{k}\cdot\mathbf{v}} d\mathbf{v}$
Dielectric function	$\epsilon(\mathbf{k},\omega) = 1 + \sum_\alpha \chi_\alpha(\mathbf{k},\omega)$
Dressing clouds described by	$\chi_\alpha(\mathbf{k},\omega)$

Plasma Line $S_{PL}(\mathbf{k},\omega)$ Ion Line $S_{IL}(\mathbf{k},\omega)$

$$S_e(\mathbf{k},\omega) = N_e \left| 1 - \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)} \right|^2 \int d\mathbf{v}\, f_e(\mathbf{v})\,\delta(\omega - \mathbf{k}\cdot\mathbf{v}) + N_i \left| \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)} \right|^2 \int d\mathbf{v}\, f_i(\mathbf{v})\,\delta(\omega - \mathbf{k}\cdot\mathbf{v})$$





electron with cloud **ion with cloud**

IS.SpecFormula3.ai

Maxwellian Plasma

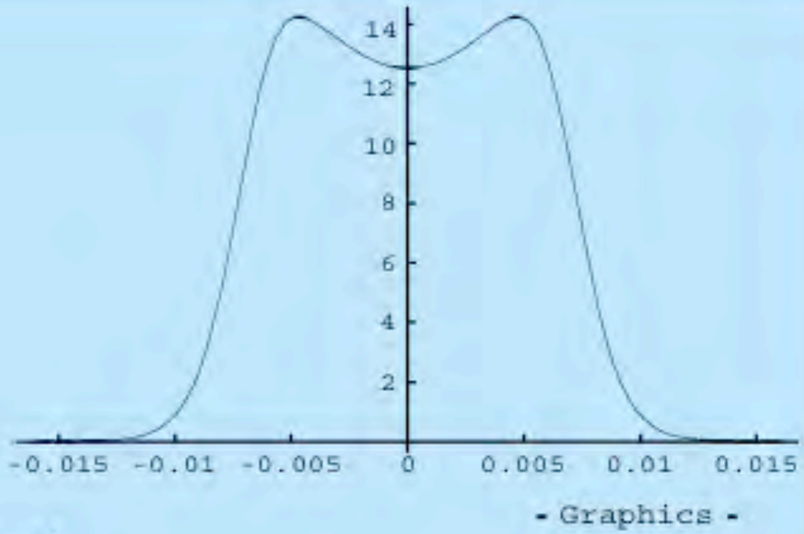
Spectrum	$S(\mathbf{k},\omega) = \sum_\alpha N_\alpha \frac{\lvert\chi_e(\mathbf{k},\omega)\rvert^2}{\lvert\epsilon(\mathbf{k},\omega)\rvert^2} \int f_\alpha(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v})d\mathbf{v}$
Maxwellian dist.	$f_\alpha(\mathbf{v}) = (\frac{m}{2\pi k_B T_\alpha})^{3/2} e^{-v^2/2k_B T_\alpha}$
Normalised freq.	$x_\alpha = \frac{v}{\sqrt{2}v_\alpha}$
Plasma Disp. Func.	$Z(x) = \frac{1}{\sqrt{\pi}} \int_{\mathcal{L}} \frac{e^{-\zeta^2/2}}{\zeta - x} d\zeta$
Susceptibility	$\chi_\alpha(\mathbf{k},\omega) = \frac{1}{(k\lambda_\alpha)^2}[1 + x_\alpha Z(x_\alpha)]$
Identity	$\int f_\alpha(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v})d\mathbf{v} = \frac{(k\lambda_\alpha)^2}{\pi\omega}\Im\chi_\alpha(\mathbf{k},\omega)$

The Imaginary Error Function $\mathrm{Erfi}(z)$

Derivative	$\frac{d}{dz}\mathrm{Erfi}(z) = \frac{2}{\sqrt{\pi}} e^{z^2}$
Relation to Error Function $\mathrm{Erf}(z)$	$\mathrm{Erfi}(z) = -i\mathrm{Erf}(iz)$
Plasma Dispersion Function	$Z(z) = \sqrt{\pi} e^{-z^2} \left[i - \mathrm{Erfi}(z)\right]$
Susceptibility	$\chi_\alpha(\mathbf{k},\omega) = \frac{1}{(k\lambda_\alpha)^2}\left[1 + x_\alpha Z(x_\alpha)\right]$
With collisions	$\chi_\alpha(\mathbf{k},\omega) = \frac{1}{(k\lambda_\alpha)^2}\left[1 + \frac{x_\alpha Z(z_\alpha)}{1 + j y_\alpha Z(z_\alpha)}\right]$

The Code

```
n = 128;
xMax = 0.1;
dx = xMax / n;
thetai = 1.0;
thetad = 1.;
pToEMass  = 1842.;
mui = 16. * pToEMass;
mud = 1.0 * pToEMass;
ri = Sqrt[thetai * mui];
rd = Sqrt[thetad * mud];
kLD2 = (0.1) ^2;
alphad = 1.;
etai = 0.95;
etai = 1.00;   (* No dust or second ion *)
w = 0. / rd;
```




```
Z[z_] := Sqrt[Pi] * E^(-(z)^2) (I - Erfi[z])
```
1

```
chiElec[z_] := (1. + (z) * Z[z]) / kLD2;
```
2

```
dielectFunc3[z_, w_] := 1 + chiElec[z] +
  thetai * etai * chiElec[ri * z] + thetad * alphad * (1 - etai) * chiElec[rd * (z - w)]
```
3

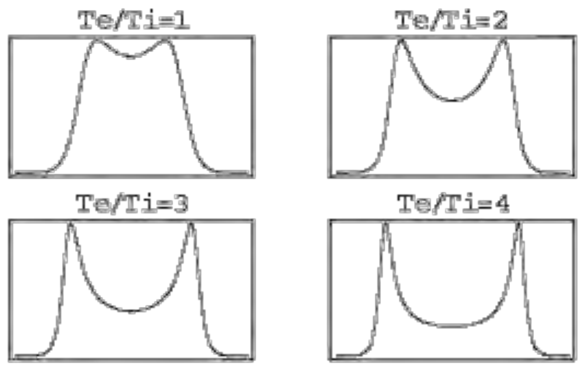
```
spec3[z_, w_] := (Abs[chiElec[z] / dielectFunc3[z, w]]) ^2 *
   (etai * (thetai * mui) ^1.5 * Exp [-z * z * ri * ri] +
     ((1. - etai) / alphad) * (thetad * mud) ^1.5 * Exp [-(z - w) * (z - w) * rd * rd])
```
4

```
spec0 = Plot[spec3[x, w], {x, -0.016, 0.016}, PlotRange → All]
```

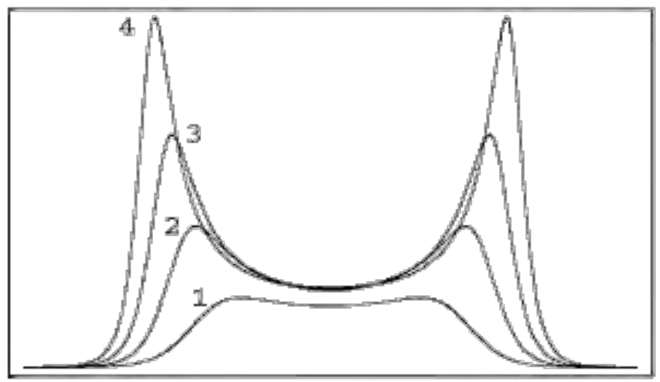
```
Block[{$DisplayFunction = Identity},
 thetai = 1.0;
 g1 = Plot[spec3[x, w], {x, -0.016, 0.016}, PlotRange → All,
   Axes -> None, Frame -> True, FrameTicks -> None, PlotLabel → "Te/Ti=1"];
 thetai = 2.0;
 g2 = Plot[spec3[x, w], {x, -0.016, 0.016}, PlotRange → All,
   Axes -> None, Frame -> True, FrameTicks -> None, PlotLabel → "Te/Ti=2"];
 thetai = 3.0;
 g3 = Plot[spec3[x, w], {x, -0.016, 0.016}, PlotRange → All,
   Axes -> None, Frame -> True, FrameTicks -> None, PlotLabel → "Te/Ti=3"];
 thetai = 4.0;
 g4 = Plot[spec3[x, w], {x, -0.016, 0.016}, PlotRange → All,
   Axes -> None, Frame -> True, FrameTicks -> None, PlotLabel → "Te/Ti=4"]]
```

```
«Graphics«
```

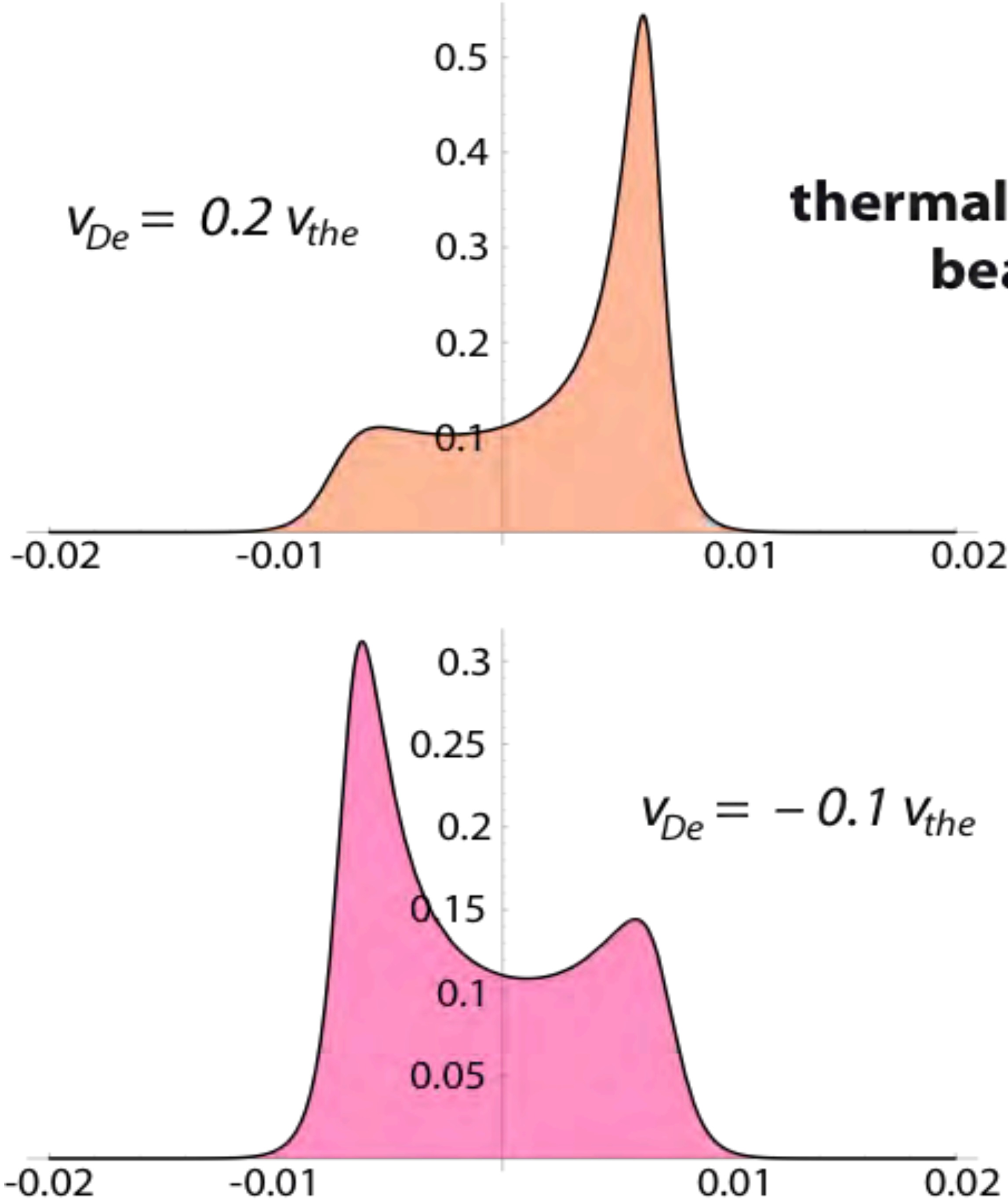
```
Show[GraphicsArray[{{g1, g2}, {g3, g4}}]];
```



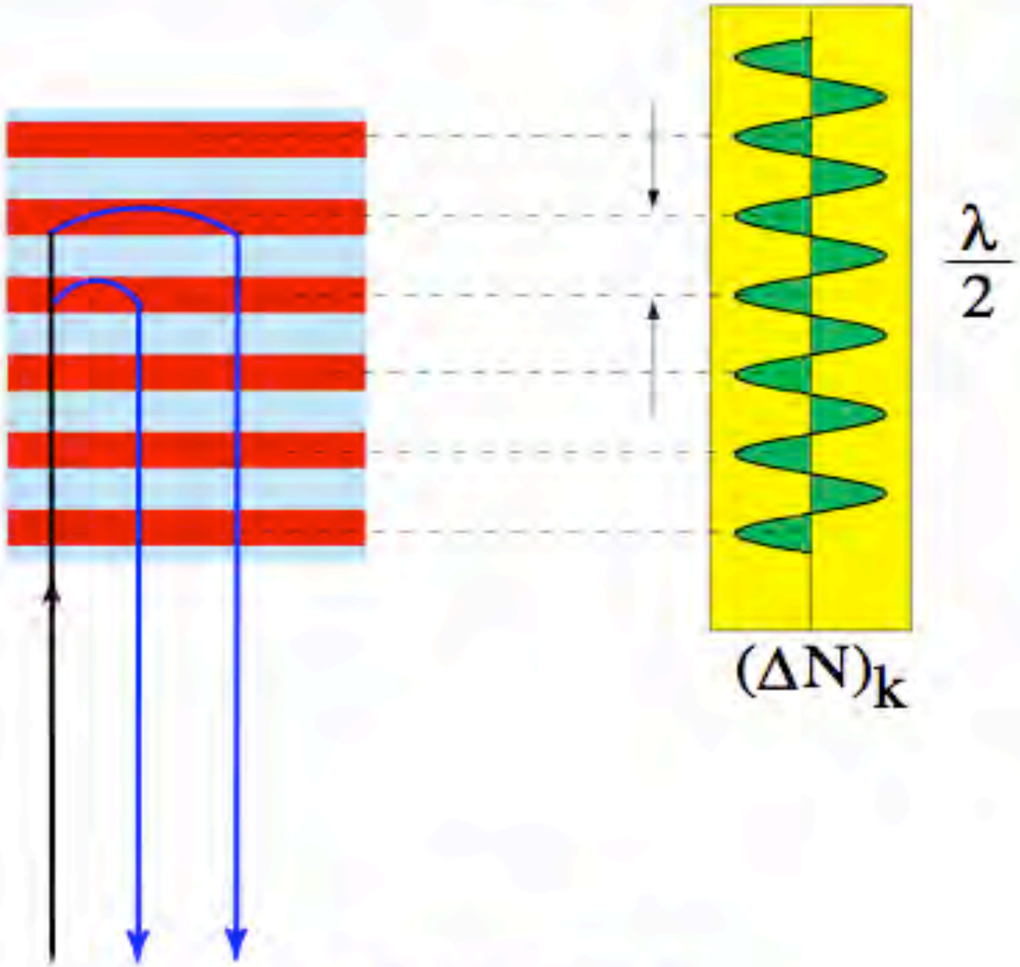

```
Show[g1, g2, g3, g4];
```



thermal electron beams



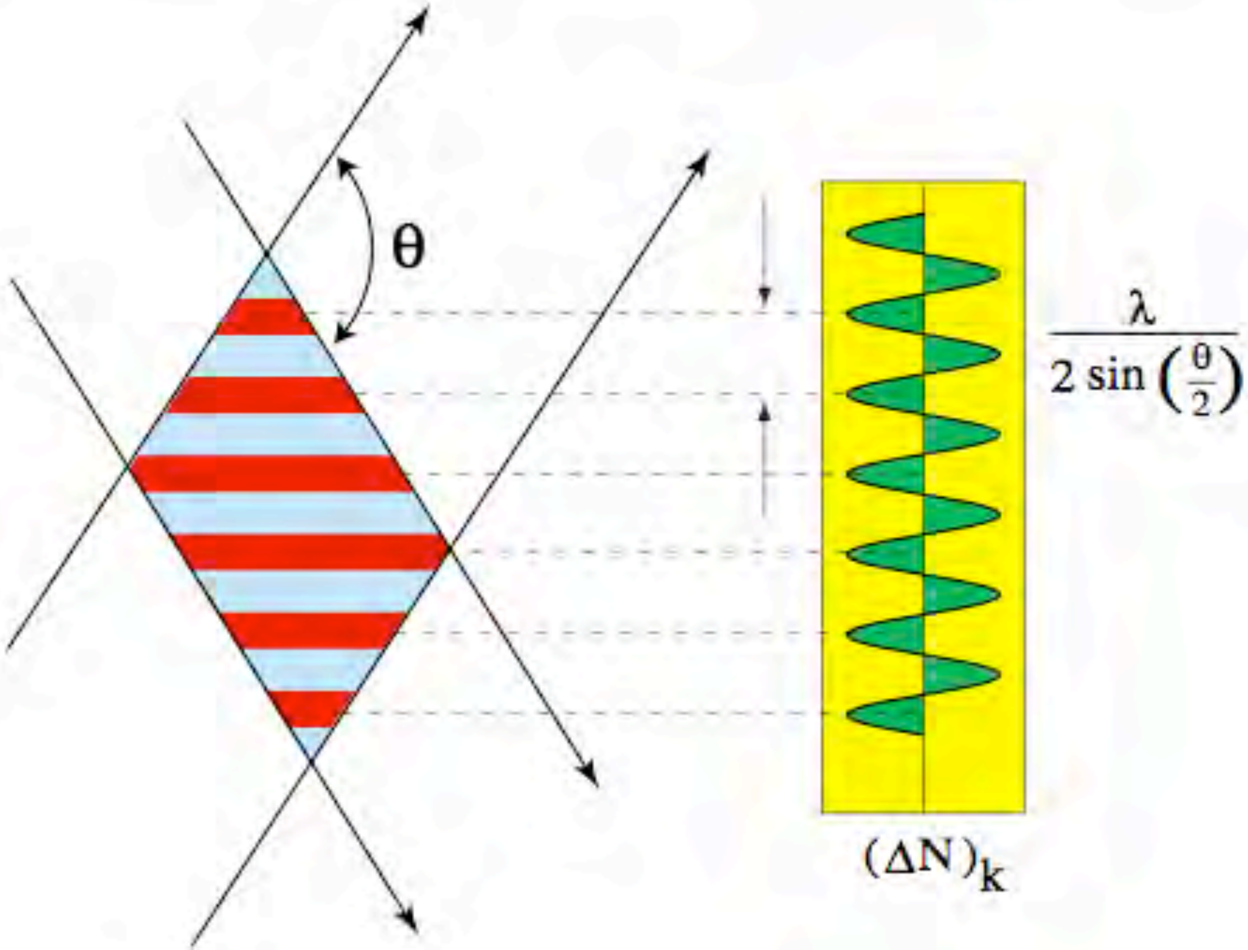
The Bragg scattering condition



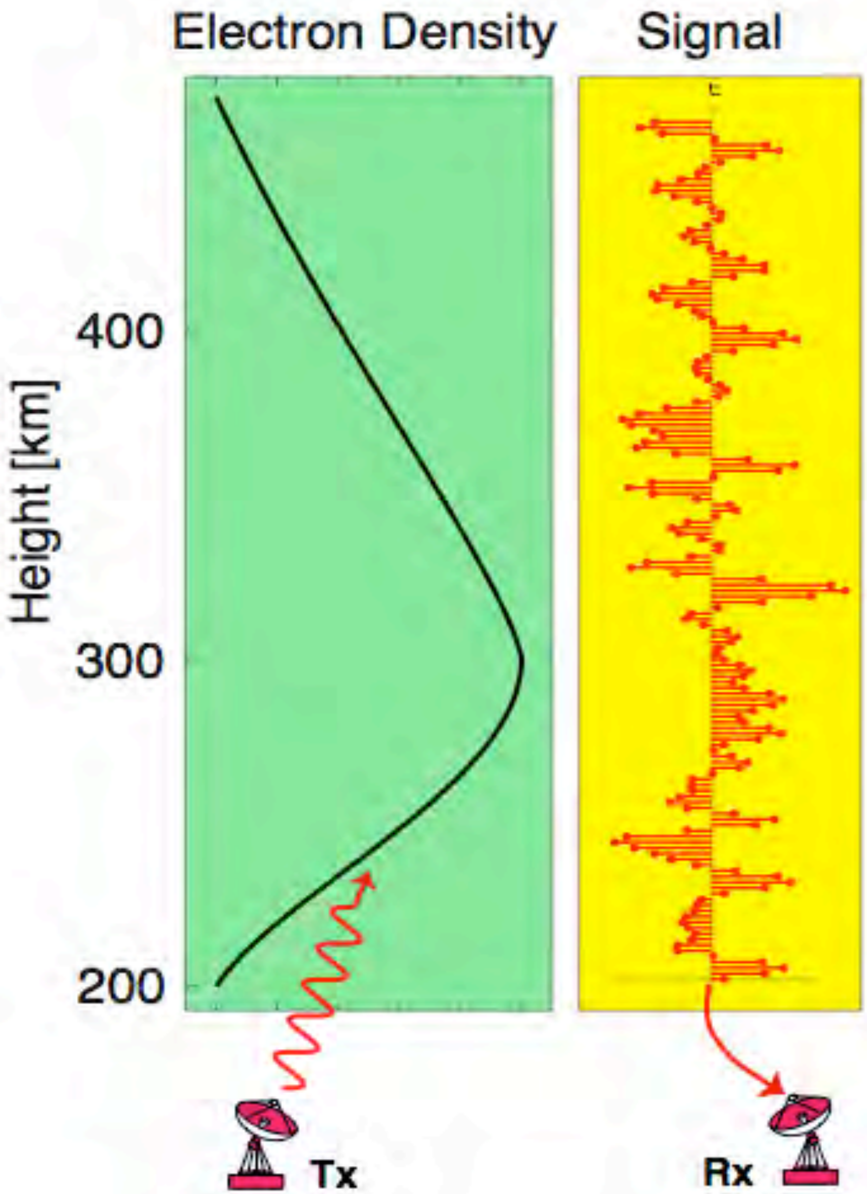
Path difference $= 1\lambda$
Both have same phase

BragCond1

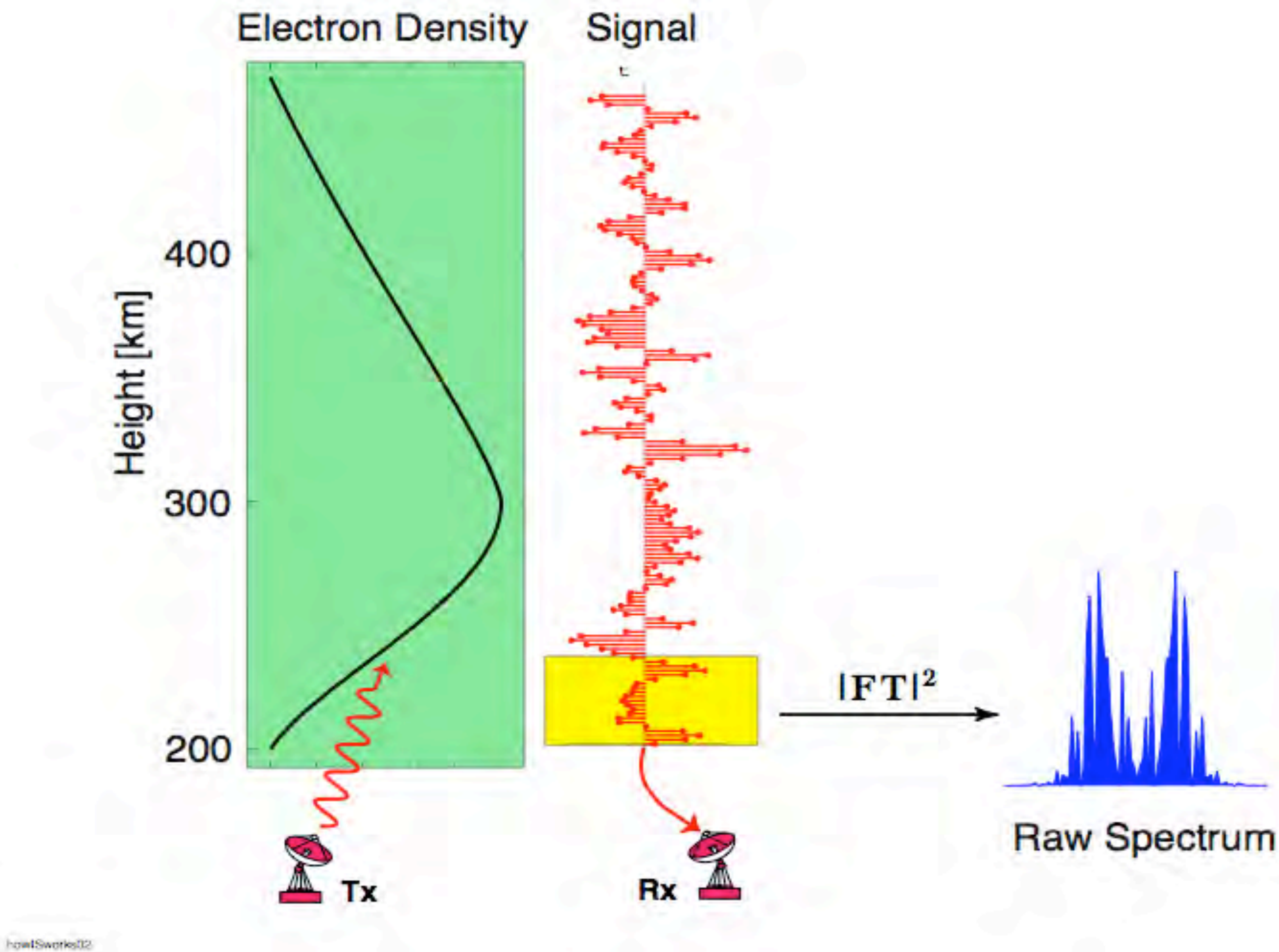
The Bragg scattering condition



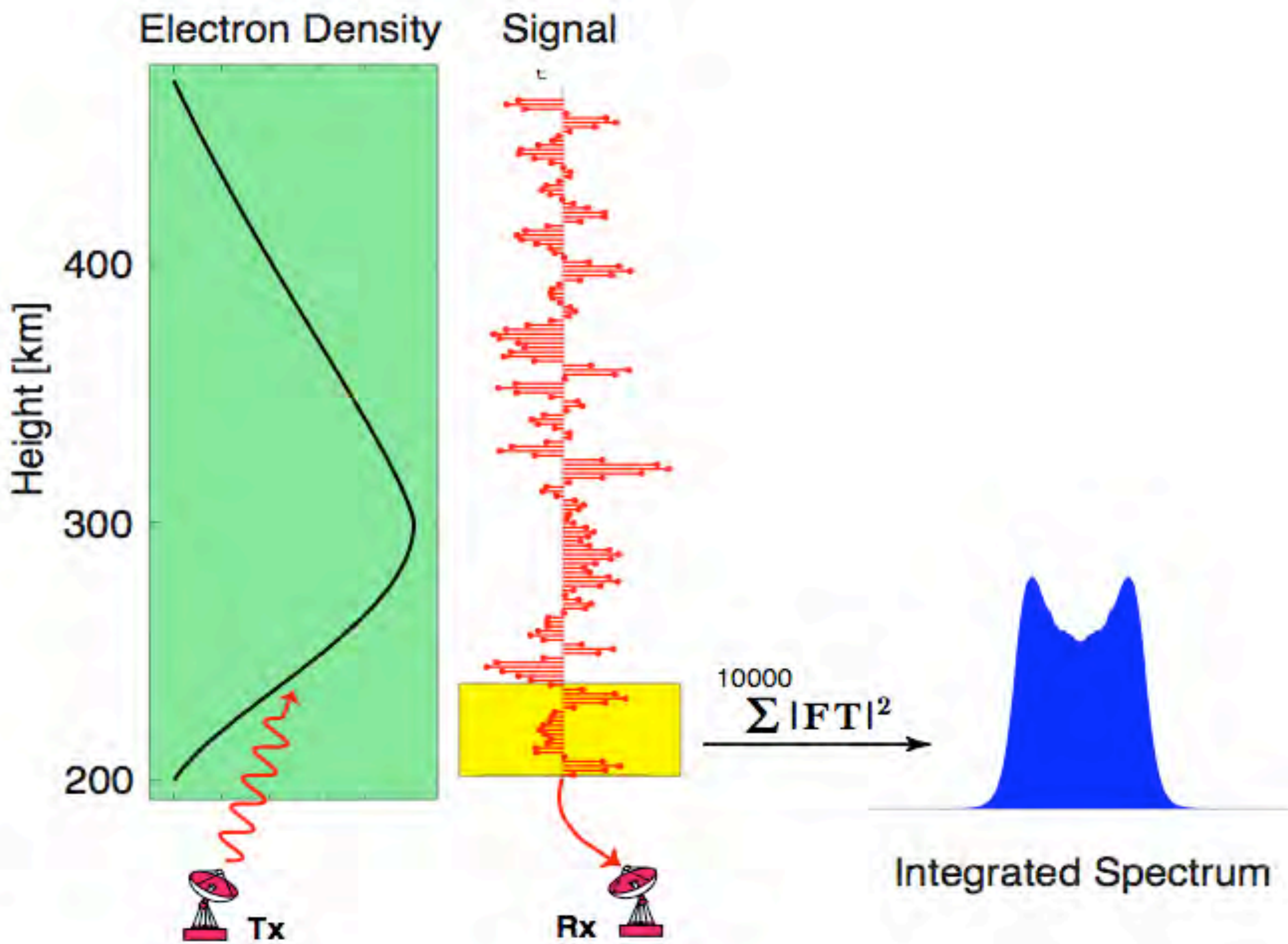
BragCond2



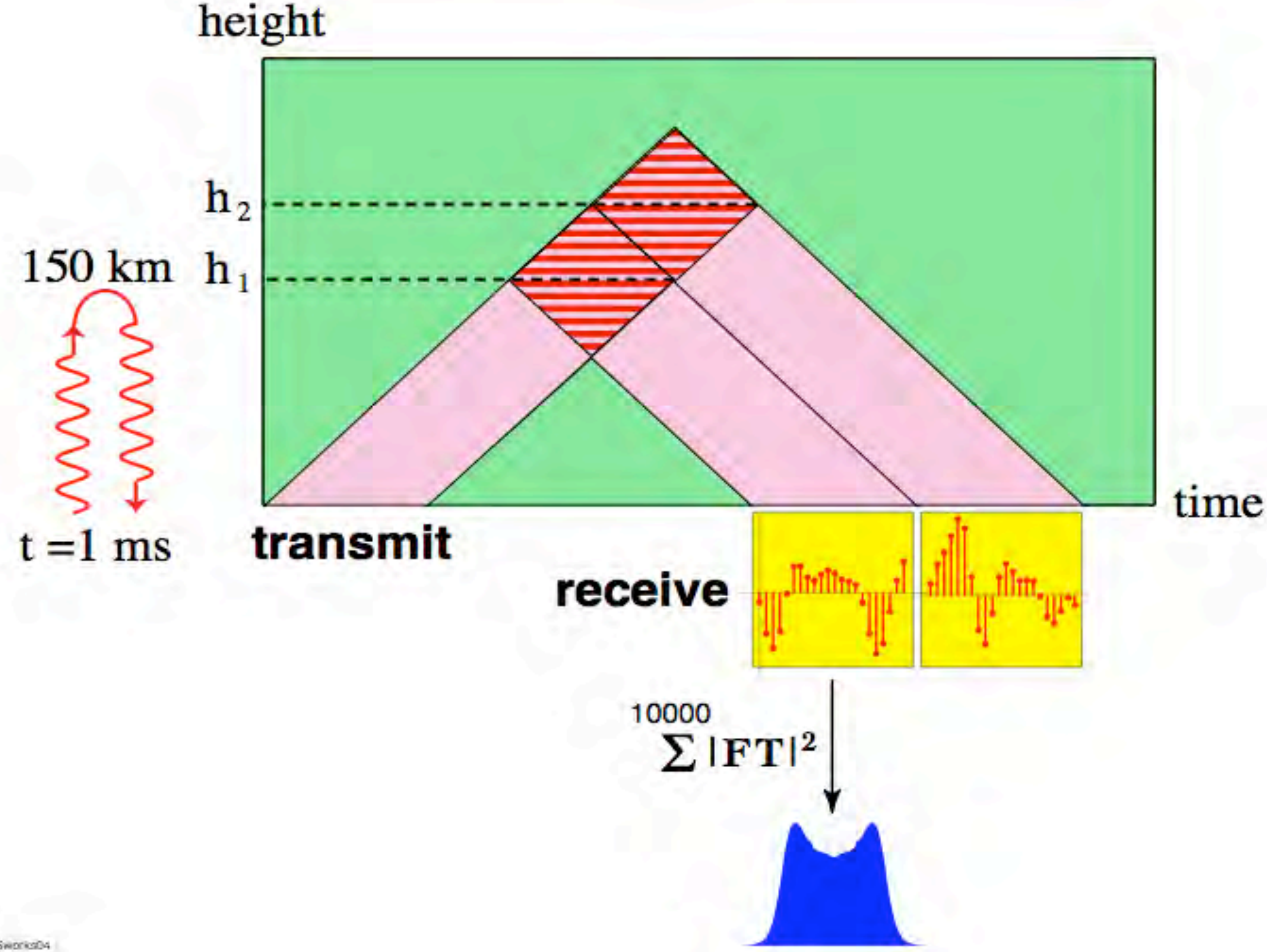
works01



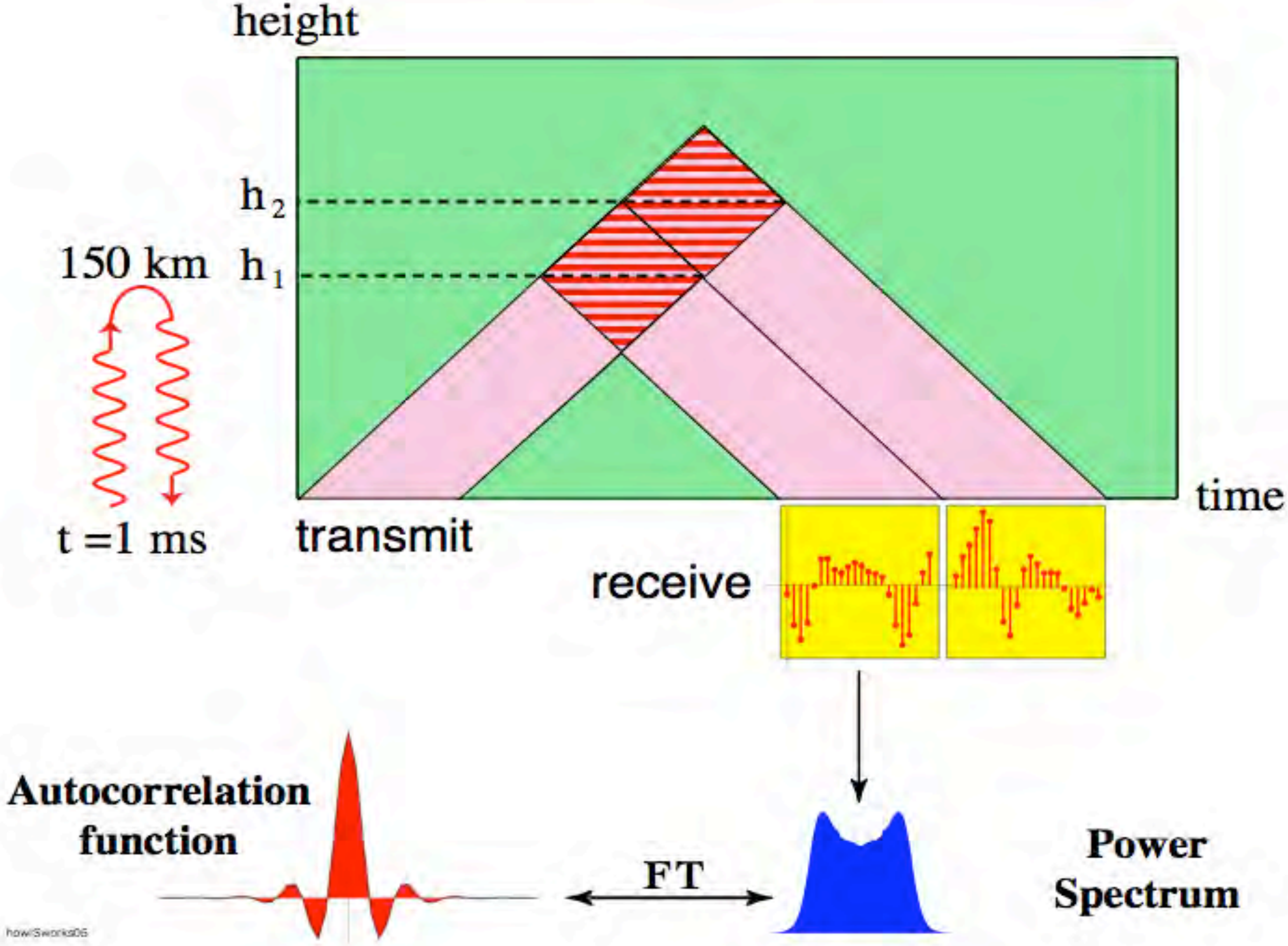
Electron Density
Signal
Height [km]
400
300
200
|FT|^2
Raw Spectrum
Tx
Rx
howISworks02



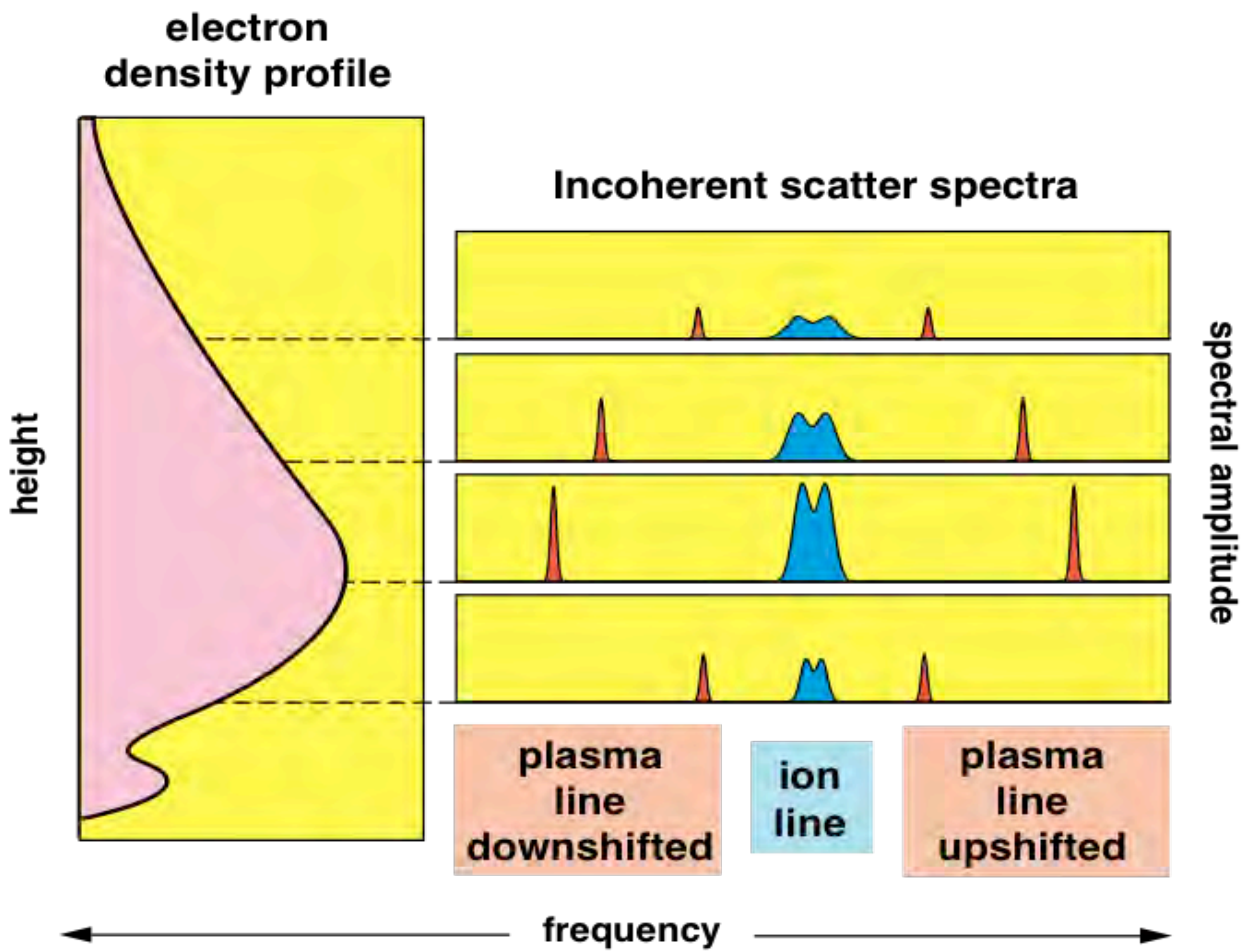
Electron Density
Signal
Height [km]
400
300
200
Tx
Rx
10000
$\sum |\mathbf{FT}|^2$
Integrated Spectrum



howISworks04

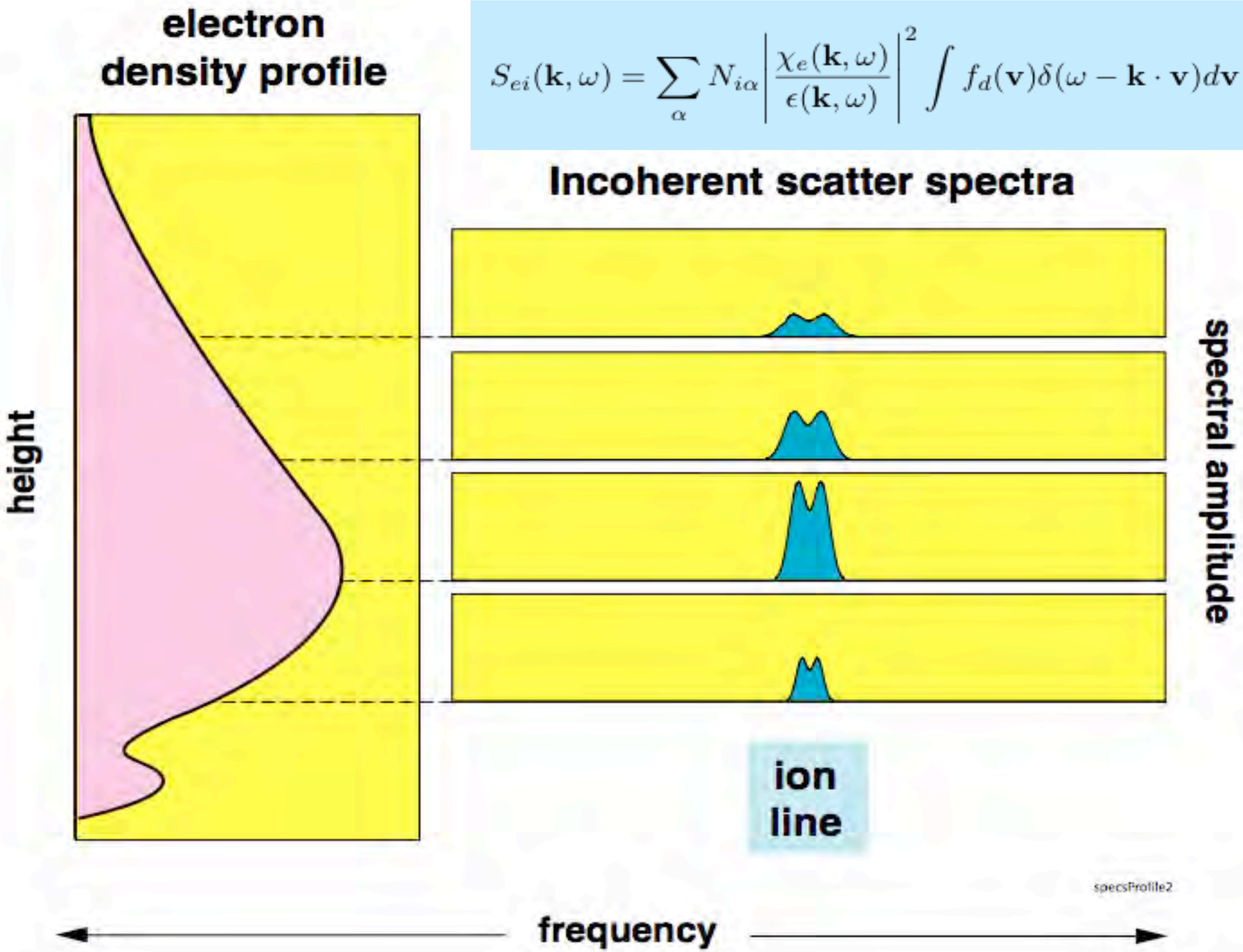


height
h 2
h 1
150 km
t =1 ms
transmit
receive
time
Autocorrelation
function
FT
Power
Spectrum
howISworks06

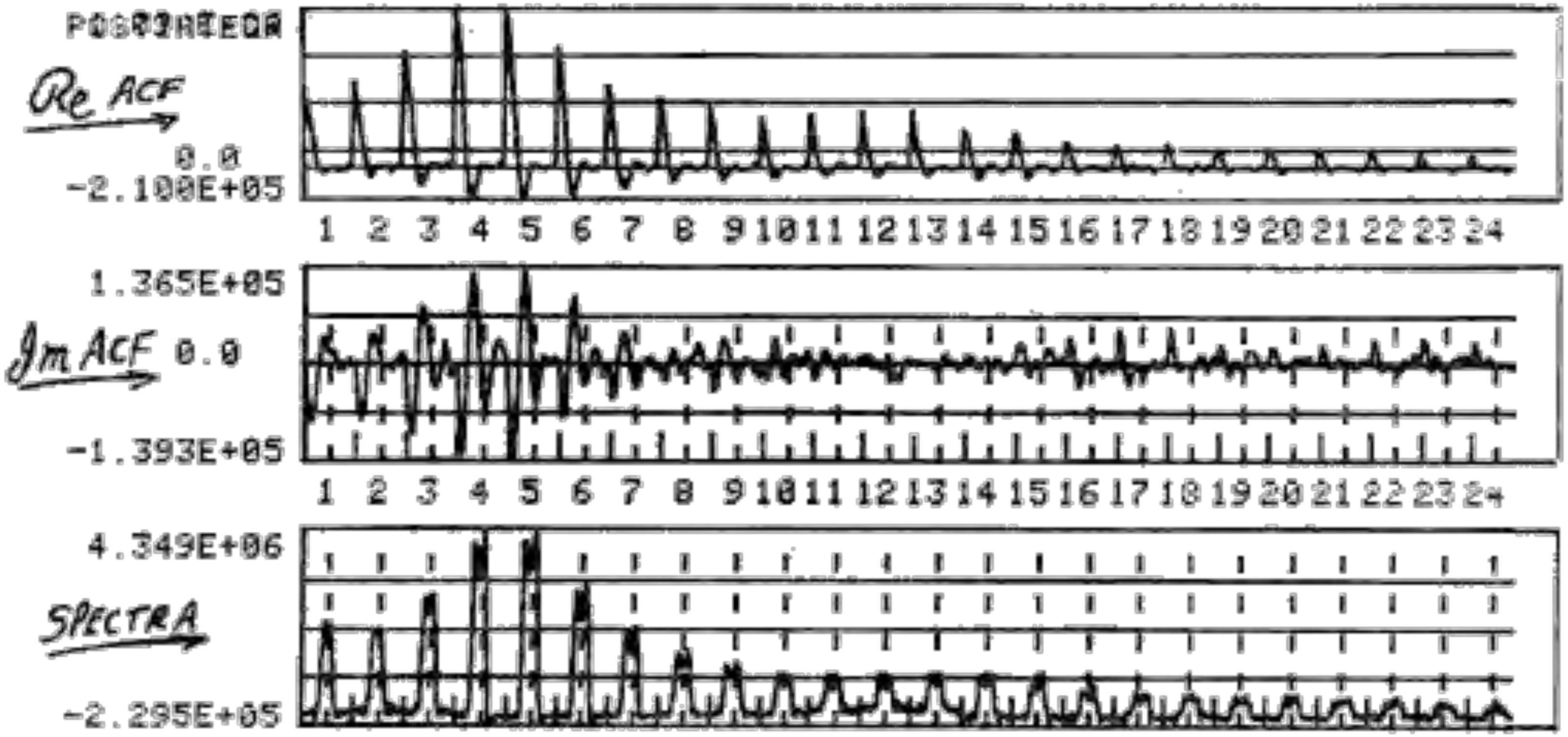


electron
density profile
Incoherent scatter spectra
height
spectral amplitude
plasma
line
downshifted
ion
line
plasma
line
upshifted
frequency

$$S_{ei}(\mathbf{k},\omega) = \sum_{\alpha} N_{i\alpha} \left| \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)} \right|^2 \int f_d(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v})d\mathbf{v}$$

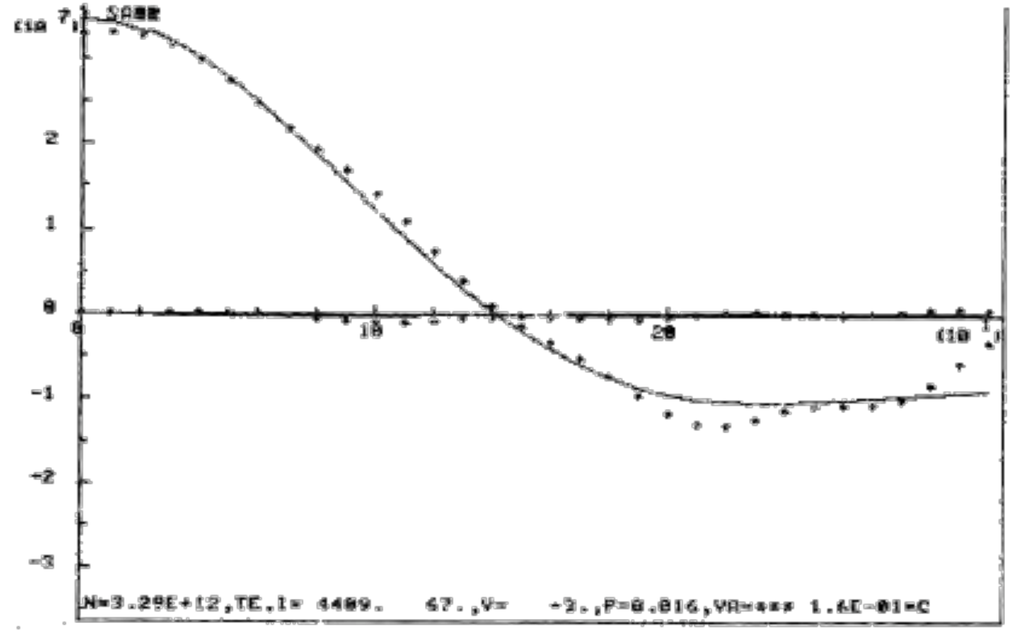


DATE:30/10-1985 TIME:17:22:28(UT) TROMSO-SITE (EXP)SP-EI-CP3SWING-
SCANS COMPUTED: 420 420 RADAR-STATUS:0000000000000010
STATUS:012003 CONTROL:010160 P-POWER:1240(KW) HV: 87(KV) TRANSMIT
MEAN OF ACFS 26 THRU 32 WAS SUBTRACTED.SYS.TEMP= 134.5 $\boxed{\text{12 POST}}$
$F_oF_2 \simeq 3\,MHz$
ANTENNA: AZ=335° EL=20° CORR-PROG:(EXP)CP-3-D-T:CCOD

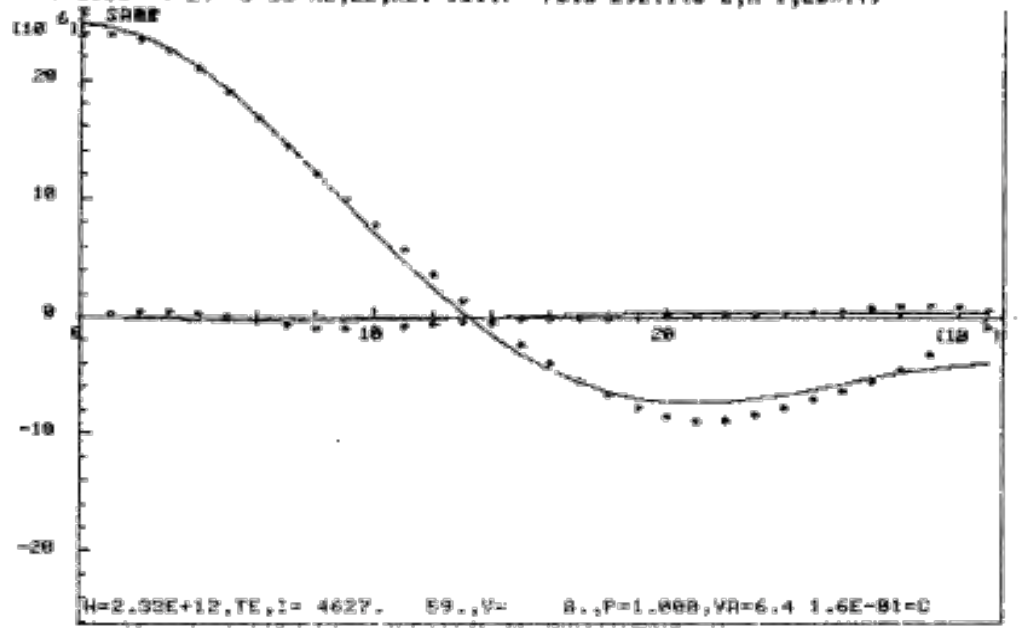


GATE NO. 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

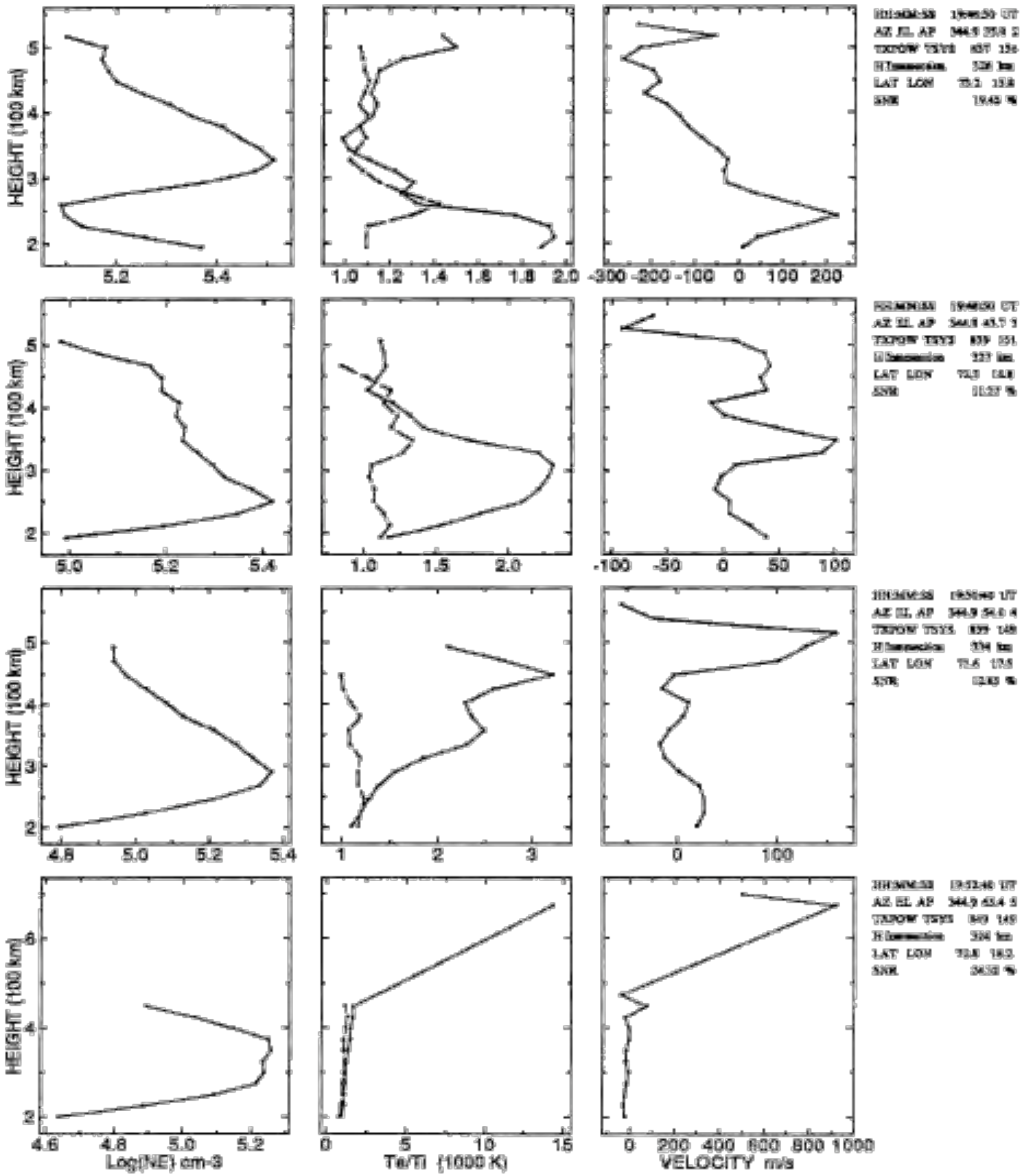
RANGE	481	544	607	670	733	796	859	922	985	1048	1111	1174	1237
ALTITUDE	180	206	232	259	286	314	342	371	400	430	460	491	521
INV. LAT.	70.5	71.0	71.5	72.0	72.5	73.0	73.5	74.0	74.5	75.0	75.5	76.0	76.4
GEOG. LAT.	73.1	73.5	73.9	74.3	74.7	75.1	75.5	75.9	76.3	76.6	77.0	77.4	77.7
GEOG. LON.	13.5	12.6	11.7	10.7	9.7	8.6	7.5	6.4	5.2	3.9	2.5	1.2	-0.24

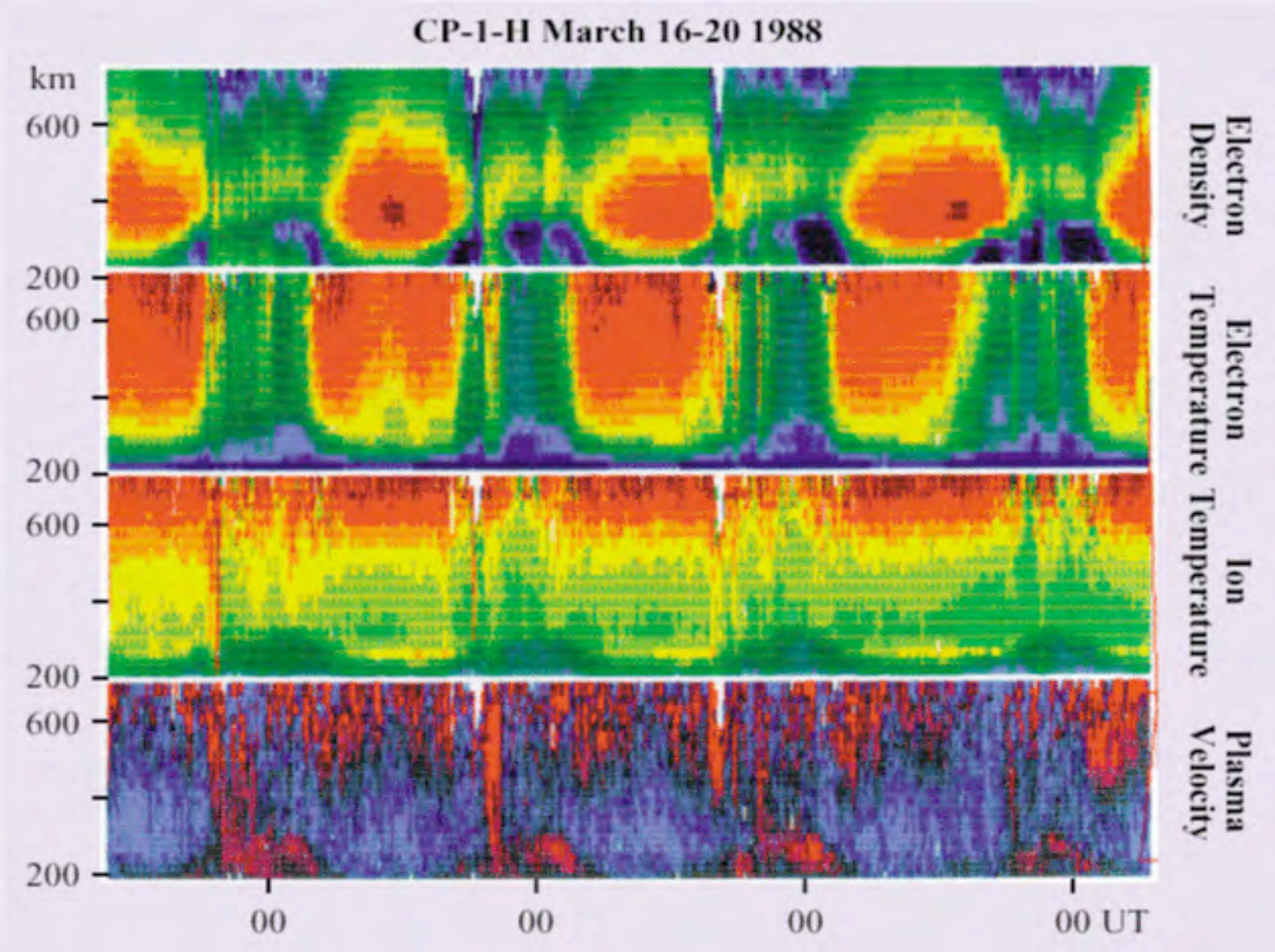


T 1986 4 24 6 58 AZ,EL,AL: 101.7 76.5 292.1(G 2,M 1,ED=TT)

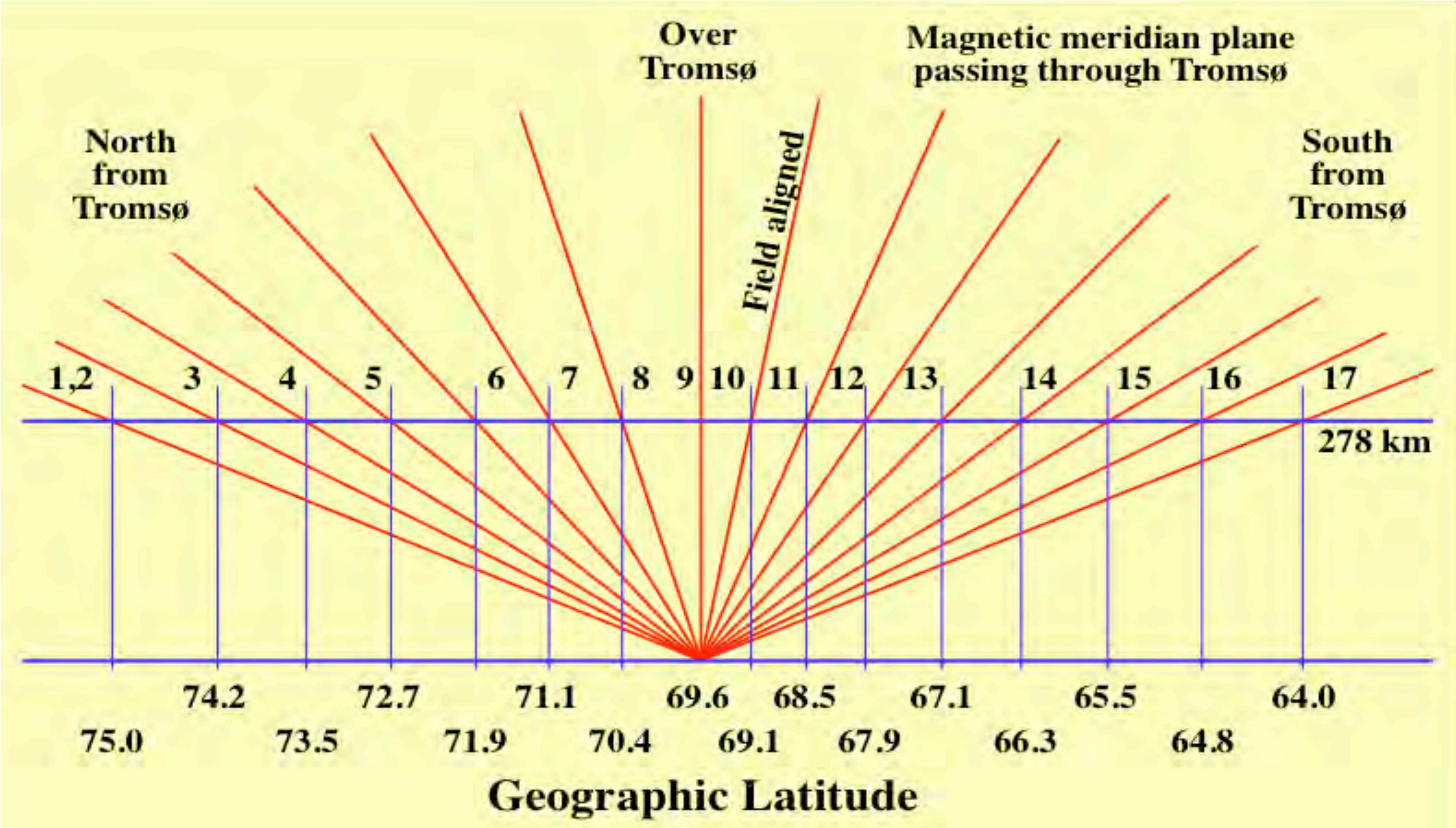


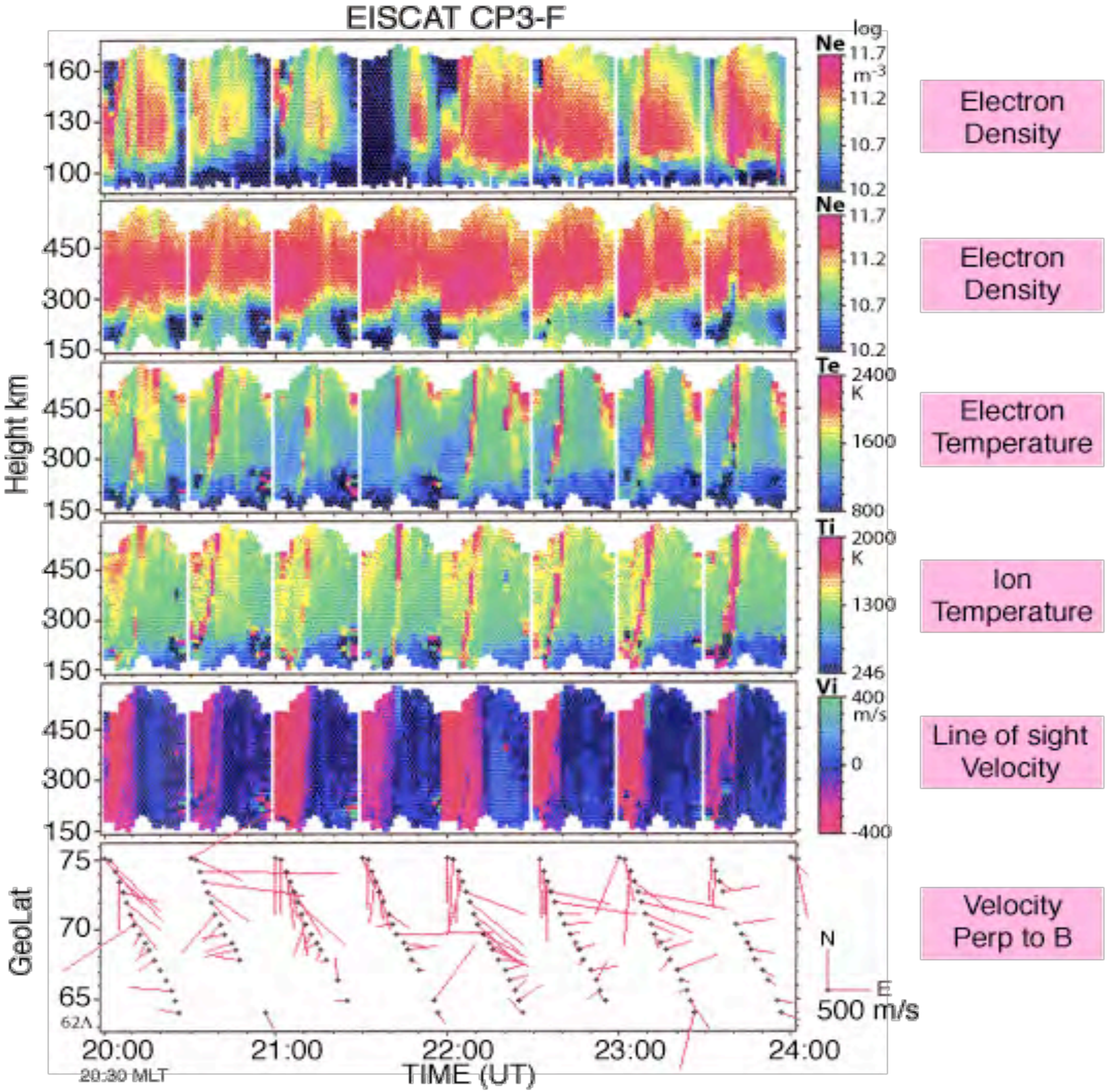
T 1986 4 24 6 58 AZ,EL,AL: 101.7 76.5 365.2(G 3,M 1,ED=TT)


HEIGHT (100 km)
Log(NE) cm-3
Te/Ti (1000 K)
VELOCITY m/s


CP-1-H March 16-20 1988
km
600
200
600
200
600
200
600
200
00
00
00
00 UT
Electron Density
Electron Temperature
Ion Temperature
Plasma Velocity

CP-3 Geometry




EISCAT CP3-F
Ne
log
11.7
m-3
11.2
10.7
10.2
Te
2400
K
1600
800
Ti
2000
K
1300
246
Vi
400
m/s
0
-400
160
130
100
450
300
150
Height km
GeoLat
75
70
65
62Λ
N
E
500 m/s
20:00
21:00
22:00
23:00
24:00
20:30 MLT
TIME (UT)
Electron Density
Electron Density
Electron Temperature
Ion Temperature
Line of sight Velocity
Velocity Perp to B

HotSpot1

Vlasov Theory of the Dielectric Function

The idea is to perturb the plasma and measure the response. The response is linearly related to the perturbation through the *dielectric function* $\epsilon(\mathbf{k}, \omega)$.

Vlasov Eq. $$\partial_t f + \mathbf{v} \cdot \partial_{\mathbf{r}} f - \frac{e}{m_e} \mathbf{E} \cdot \partial_{\mathbf{v}} f = 0$$

Linear pert. $$\delta f(\mathbf{k}, \mathbf{v}, \omega) = -\frac{(e/m_e)\mathbf{E} \cdot \partial_{\mathbf{v}} f(\mathbf{v})}{i(\omega - \mathbf{k} \cdot \mathbf{v})}$$

Densities $$\rho = \delta\rho + \rho_s, \quad \delta\rho = N \int \delta f d\mathbf{v}$$

Poisson Eq. $$-i\mathbf{k} \cdot \mathbf{E} = \frac{\delta\rho}{\varepsilon_o} + \frac{\rho_s}{\varepsilon_o}$$

Result $$-i\mathbf{k} \cdot \mathbf{E} \left[1 + \frac{\omega_p^2}{k^2} \int_{\mathcal{L}} \frac{\mathbf{k} \cdot \partial_{\mathbf{v}} f(\mathbf{v})}{\omega - \mathbf{k} \cdot \mathbf{v}} d\mathbf{v} \right] = \frac{\rho_s}{\varepsilon_o}$$

Therefore $$-i\mathbf{k} \cdot \mathbf{E} = \frac{\rho_s}{\varepsilon_o \epsilon(\mathbf{k}, \omega)}$$

Where $$\epsilon(\mathbf{k}, \omega) = 1 + \frac{\omega_p^2}{k^2} \int_{\mathcal{L}} \frac{\mathbf{k} \cdot \partial_{\mathbf{v}} f(\mathbf{v})}{\omega - \mathbf{k} \cdot \mathbf{v}} d\mathbf{v}$$

Illustration: The Free Electron Gas

Bare electron j	$\rho_{ej}^{(b)}(\mathbf{r},t) = q_e\delta(\mathbf{r}-\mathbf{r}_j(t))$
Unperturbed trajectory	$\mathbf{r} = \mathbf{r}_{jo} + \mathbf{v}_j t$
Fourier transform	$\rho_{ej}^{(b)}(\mathbf{k},\omega) = q_e e^{-i\mathbf{k}\cdot\mathbf{r}_{jo}}\delta(\omega-\mathbf{k}\cdot\mathbf{v}_j)$
Dressed electron	$\rho_{ej}(\mathbf{k},\omega) = \dfrac{q_e e^{-i\mathbf{k}\cdot\mathbf{r}_{jo}}\delta(\omega-\mathbf{k}\cdot\mathbf{v}_j)}{\epsilon(\mathbf{k},\omega)}$
Dielectric function	$\epsilon(\mathbf{k},\omega) = 1 + \chi_e(\mathbf{k},\omega)$
Susceptibility	$\chi_e(\mathbf{k},\omega) = \dfrac{\omega_{pe}^2}{k^2}\displaystyle\int_{\mathcal{L}} \dfrac{\mathbf{k}\cdot\partial_{\mathbf{v}} f_e(\mathbf{v})}{\omega-\mathbf{k}\cdot\mathbf{v}}d\mathbf{v}$
Total density	$\rho_e(\mathbf{k},\omega) = \displaystyle\sum_j q_e e^{-i\mathbf{k}\cdot\mathbf{r}_{jo}}\delta(\omega-\mathbf{k}\cdot\mathbf{v}_j)$
Power Spectrum	$S(\mathbf{k},\omega) = \langle\rho_e(\mathbf{k},\omega)\rho_e^*(\mathbf{k},\omega)\rangle$

...The Free Electron Gas

Power Spectrum $$S(\mathbf{k},\omega) = \frac{q_e^2\left\langle\sum_i\sum_j \delta(\omega-\mathbf{k}\cdot\mathbf{v}_i)\delta(\omega-\mathbf{k}\cdot\mathbf{v}_j)\right\rangle}{|\epsilon(\mathbf{k},\omega)|^2}$$

Uncorrelated dressed particles, $i \neq j$

$$\left\langle\delta(\omega - \mathbf{k}\cdot\mathbf{v}_i)\delta(\omega - \mathbf{k}\cdot\mathbf{v}_j)\right\rangle = 0$$

Result (per electron) $$S(\mathbf{k},\omega) = \frac{\left\langle\delta(\omega - \mathbf{k}\cdot\mathbf{v})\right\rangle}{|\epsilon(\mathbf{k},\omega)|^2}$$

The ensemble average $$S(\mathbf{k},\omega) = \frac{\int f_e(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v})d\mathbf{v}}{|\epsilon(\mathbf{k},\omega)|^2}$$

Maxwellian e-gas $$\int f_e(\mathbf{v})\delta(\omega - \mathbf{k}\cdot\mathbf{v})d\mathbf{v} = \frac{(k\lambda_e)^2}{\pi\omega}\Im\epsilon(\mathbf{k},\omega)$$

Use of the identity $$\frac{\Im\epsilon}{|\epsilon(\mathbf{k},\omega)|^2} = \Im\frac{1}{\epsilon(\mathbf{k},\omega)}$$

Gives $$S(\mathbf{k},\omega) = \frac{(k\lambda_e)^2}{\pi\omega}\Im\frac{1}{\epsilon(\mathbf{k},\omega)}$$

This is precisely the result from the
fluctuation dissipation theorem

δN_e due to a Test Particle: Vlasov Theory

From the kinetic Vlasov equation for electrons

$$\partial_t f + \mathbf{v} \cdot \partial_{\mathbf{r}} f - \frac{e}{m_e} \mathbf{E} \cdot \partial_{\mathbf{v}} f = 0$$

one obtains the perturbation of the electron distribution function due to the fluctuation electric field of a dressed test particle j of species α

$$\delta f_{e\alpha j}(\mathbf{k}, \mathbf{v}, \omega) = -\frac{(e/m_e)\mathbf{E}_{\alpha j} \cdot \partial_{\mathbf{v}} f(\mathbf{v})}{i(\omega - \mathbf{k} \cdot \mathbf{v})}$$

From Poisson equation one finds the electric field fluctuation due to the *dressed* test particle

$$\mathbf{E}_{\alpha j}(\mathbf{k}, \omega) = \frac{i\mathbf{k}}{k^2 \varepsilon_o \epsilon(\mathbf{k}, \omega)} \mathcal{Z}_{\alpha j} q_e \delta(\omega - \mathbf{k} \cdot \mathbf{v}_j) e^{-i\mathbf{k} \cdot \mathbf{r}_{jo}}$$

The electron number density fluctuation is found by integrating over the distribution function

$$\delta N_{e\alpha j}(\mathbf{k}, \omega) = N_e \int \delta f_{e\alpha j}(\mathbf{k}, \mathbf{v}, \omega) d\mathbf{v}$$

Employing the first two equations in the last equation it is easy to find

$$\delta N_{e\alpha j}(\mathbf{k}, \omega) = \frac{\chi_e(\mathbf{k}, \omega)}{\epsilon(\mathbf{k}, \omega)} \mathcal{Z}_{\alpha j} \delta(\omega - \mathbf{k} \cdot \mathbf{v}_j) e^{-i\mathbf{k} \cdot \mathbf{r}_{jo}}$$

The next steps go identical as in the free electron plasma.

Multicomponent Plasma

After using the Vlasov equation to take account of the electron-ion interactions, the **electron density fluctuations** in the ω-$\mathbf{k}$ space are (**not** power spectrum):

$$\rho_{ee}(\mathbf{k},\omega) = \left[1 - \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}\right]\rho_e^{(b)}(\mathbf{k},\omega)$$

$$\rho_{e\alpha}(\mathbf{k},\omega) = \frac{\chi_e(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)}\rho_\alpha^{(b)}(\mathbf{k},\omega)$$

where the indeces ee mean the electron fluctuations due to electrons, and $e\alpha$ the electron fluctuation due to ions of species α.

The screening of the bare particles can be considered as a **renormalization** of the particle's charge in the usual sense of field theories. The renormalization is determined by the linear polarization response functions, that is, the susceptibilities χ of the plasma components. The screened particles are, again in the sense of field theories, **quasi-particles**

Since dust incoherent scattering is due only to electrons, we are interested in the electron fluctuations due to **charged dust particles** when $\alpha = d$.